

LIMITED SCOPE ADIC TRANSFORMATIONS

SARAH BAILEY FRICK

THE OHIO STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS, 100 MATH
TOWER, 231 W. 18TH AVE., COLUMBUS, OH 43210

ABSTRACT. We introduce a family of adic transformations on diagrams that are nonstationary and nonsimple. This family includes some previously studied adic transformations. We relate the dimension group of each these diagrams to the dynamical system determined by the adic transformation on the infinite edge paths, and we explicitly compute the dimension group for a subfamily. We also determine the ergodic adic invariant probability measures for this subfamily, and show that each system of the subfamily is loosely Bernoulli. We also give examples of particular adic transformations with roots of unity as well as one which is totally ergodic called the Euler adic. We also show that the Euler adic is loosely Bernoulli.

1. INTRODUCTION

In 1972, Bratteli introduced infinite directed graphs known now as *Bratteli diagrams* as a tool to study approximately finite-dimensional (AF) algebras, [?]. To each of these graphs, Vershik introduced a transformation, now known as the *Bratteli-Vershik* or *adic* transformation as a method of modeling cutting and stacking transformations, [?, ?, ?]. He also showed that every ergodic measure-preserving transformation on a Lebesgue space is isomorphic to a Bratteli-Vershik transformation which has a unique invariant measure associated to it. Herman, Putnam, and Skau went on to show that every minimal homeomorphism of the Cantor set is topologically conjugate to a adic transformation with certain properties, [?].

One adic transformation proposed by Vershik is the Pascal adic transformation. This single transformation has been the subject of much study; see [?, ?, ?, ?, ?] and the references they contain. In this paper we study a specific class of Bratteli-Vershik transformations known as *limited scope adic transformations* which have zero entropy and contains all of the transformations contained in [?], including the Pascal adic transformation. We show that the dimension group associated to the Bratteli diagram on which the limited scope adic are defined is order isomorphic to the continuous functions from the infinite path space into the integers modulo the continuous coboundaries. This is an extension of the result of Herman, Putnam, and Skau for minimal Cantor systems, [?]. We also compute the dimension group directly for a particular subclass of limited scope adic transformations which are associated to polynomials over the natural numbers, defined below. The transformations contained in [?], including the Pascal adic, are contained in this subclass. Different limited scope adic transformations may have very different dynamical properties, and we establish several dynamical properties for the subclass of limited scope adic transformations associated to polynomials over the natural numbers as well as for the Euler adic transformation, which is defined below. In

particular we determine all of the ergodic adic invariant measures for the subclass of limited scope adic transformations associated to polynomials over the natural numbers. We show that many of the adic transformations associated to polynomials over the natural numbers have roots of unity as eigenvalues whereas the Euler adic transformation is totally ergodic. We conclude by showing that all of the adic transformations associated to polynomials over the natural numbers as well as the Euler adic transformation is loosely Bernoulli. This paper is based on the Ph.D. dissertation of the author at the University of North Carolina at Chapel Hill under the supervision of Karl Petersen, [?].

Let $(\mathcal{V}, \mathcal{E})$ be a Bratteli diagram such that for a constant $d \in \mathbb{N}$, the number of vertices at level n is $nd + 1$ and each vertex, labeled (n, k) , $0 \leq k \leq dn$, is connected by some positive number of edges to each vertex in $(n + 1, k + i)$ for all $i \in \{0, 1, 2, \dots, d\}$, and there are no edges elsewhere. We denote this family of Bratteli diagrams by $\mathcal{D}_{\mathcal{L}}$.

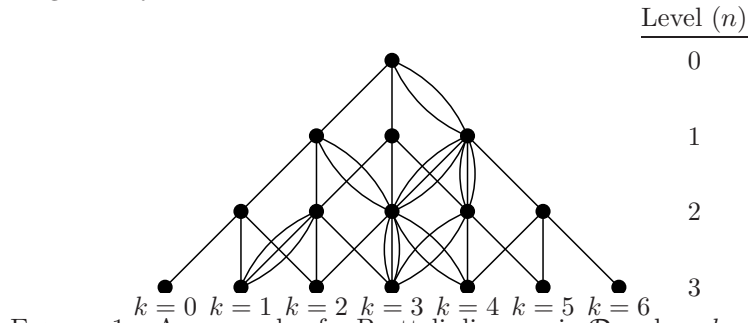


FIGURE 1. An example of a Bratteli diagram in $\mathcal{D}_{\mathcal{L}}$ when $d = 2$.

For any diagram $(\mathcal{V}, \mathcal{E}) \in \mathcal{D}_{\mathcal{L}}$, X is the space of infinite edge paths on $(\mathcal{V}, \mathcal{E})$. The vertex through which γ passes at level n is denoted $(n, k_n(\gamma))$. X is a metric space with the standard metric: for $\gamma = \gamma_0\gamma_1\dots$ and $\xi = \xi_0\xi_1\dots$, $d(\gamma, \xi) = 2^{-j}$, where $j = \inf\{j_j \neq \xi_j\}$.

An ordering given to edges of the diagram which terminate into the same vertex is extended to a partial ordering on the entire path space. Two paths γ and ξ are *comparable* if they agree after some level n and disagree on level $n - 1$. We then define $\gamma < \xi$ if and only if $\gamma_{n-1} < \xi_{n-1}$ with respect to the edge ordering. When we have endowed a Bratteli diagram $(\mathcal{V}, \mathcal{E})$ with an edge ordering extending to infinite paths, we say that $(\mathcal{V}, \mathcal{E})$ is an ordered Bratteli diagram and denote it by $(\mathcal{V}, \mathcal{E}, \geq)$. The diagrams in $\mathcal{D}_{\mathcal{L}}$ are drawn so that edges with the same range increase in order from left to right.

We denote by X_{\max} the set of paths in X for which all edges are maximal with respect to their edge ordering. Likewise we denote by X_{\min} the set of paths in X for which all edges are minimal with respect to their edge ordering. For Bratteli diagrams in $\mathcal{D}_{\mathcal{L}}$ there are a countable number of paths in $X_{\max} \cup X_{\min}$. Indeed, for every k in the set $\{0, 1, \dots\} \cup \{\infty\}$ there is a unique associated path in X_{\max} , denoted γ_{\max}^k , which is defined as follows. For $k \neq \infty$ γ_{\max}^k is the path in X that travels down the far right side of the graph, following maximal edges, to level $n_0 - 1$, where $n_0 \in \mathbb{N}$ is such that $(n_0 - 1)d < k \leq n_0d$, and then connects to vertex (n_0, k) along the maximal edge. Then for $n \geq n_0$, $k_n(\gamma_{\max}^k) = k$, and γ_{\max}^k follows a maximal edge. The path γ_{\max}^∞ is the path which travels through the vertices (n, dn) along maximal edges for all $n \in \mathbb{N}$. Likewise for every k in the set $\{0, 1, \dots\} \cup \{\infty\}$

there is a unique path in X_{\min} denoted γ_{\min}^k . For $k \neq \infty$ this is the path in X that travels down the left side of the graph along minimal edges to level $n_0 - 1$, where $n_0 \in \mathbb{N}$ is such that $(n_0 - 1)d < k \leq n_0 d$, and then connects to vertex $(n_0, n_0 d - k)$ along the minimal edge. Then for $n \geq n_0$, $k_n(\gamma_{\min}^k) = nd - k$. The path γ_{\min}^∞ is the path which travels through the vertices $(n, 0)$ along minimal edges for all $n \in \mathbb{N}$.

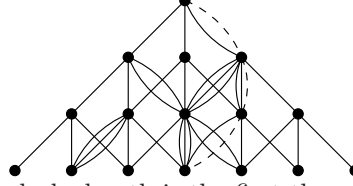


FIGURE 2. The dashed path is the first three edges of γ_{\max}^3 .

Let $T : X \rightarrow X$ be the Bratteli-Vershik transformation which maps a path γ to the next largest path with respect to the partial ordering of edges, if one exists. If such a path exists, we will call it the *successor* of γ . All paths in $X \setminus X_{\max}$ have unique successors, and hence T is well defined off of X_{\max} . We will now define T on X_{\max} so that $T(\gamma_{\max}^k) = \gamma_{\min}^k$ for $0 < k < \infty$, $T(\gamma_{\max}^0) = \gamma_{\min}^\infty$, and $T(\gamma_{\min}^\infty) = \gamma_{\max}^0$. In this way T is a bijection on the whole space X ; but not continuous on X_{\max} . The family of Bratteli-Vershik systems determined in the manner from Bratteli diagrams in $\mathcal{D}_{\mathcal{L}}$ are said to be of limited scope and will be denoted $\mathcal{S}_{\mathcal{L}}$.

For $(X, T) \in \mathcal{S}_{\mathcal{L}}$, we say a path $\gamma \in X$ is *eventually diagonal to the left* if there exists and $N \geq 0$ such that for $n \geq N$, $k_n(\gamma) = k_N(\gamma)$. We say a path $\gamma \in X$ is *eventually diagonal to the right* if there exists an $M \geq 0$ such that for $m \geq M$, $k_m(\gamma) = dm - k_M(\gamma)$. We will say that a path is *eventually diagonal* if the direction is either clear or unknown. All paths in the orbits of X_{\max} and X_{\min} are eventually diagonal.

Proposition 1.1. *For every $\gamma \in X$, exactly one of the following holds.*

1. γ is eventually diagonal to the right.
2. γ is eventually diagonal to the left.
3. $\mathcal{O}(\gamma) = X$

Proof. If γ is not eventually diagonal, both $k_n(\gamma)$ and $dn - k_n(\gamma)$ are unbounded. Then for any $\xi \in X$ and $m \in \mathbb{N}$ there is an $n_0 > m$ such that $k_m(\xi) \leq k_{n_0}(\gamma)$ and $dm - k_m(\xi) \leq dn_0 - k_{n_0}(\gamma)$. Hence, $k_{n_0}(\gamma) - d(n_0 - m) \leq k_m(\xi) \leq k_{n_0}(\gamma)$. Therefore there is a path from $(m, k_m(\xi))$ to $(n_0, k_{n_0}(\gamma))$. Then there is a $j \in \mathbb{Z}$ so that $T^j \gamma$ coincides with ξ along the first m edges, showing that $\mathcal{O}(\gamma)$ is dense in X .

If γ is eventually diagonal to the right (resp. to the left), there exists an $N \in \mathbb{N}$ such that for any $\eta \in \mathcal{O}(\gamma)$ and all $n \in \mathbb{N}$, $k_n(\eta) < N$ or $k_n(\eta) > dn - N$. Now choose $\xi \in X$ and $m \in \mathbb{N}$ for which $N < k_m(\xi) < dm - N$ and let $B_{2^{-m}}(\xi)$ be the ball of radius 2^{-m} around ξ . Then $\mathcal{O}(\gamma) \cap B_{2^{-m}}(\xi) = \emptyset$. Hence, $\mathcal{O}(\gamma)$ is not dense in X . \square

2. EXAMPLES

In this section we give a description of some examples of Bratteli-diagrams in $\mathcal{D}_{\mathcal{L}}$. Endowed with the above adic transformation they generate Bratteli-Vershik

systems in $\mathcal{S}_{\mathcal{L}}$. Specifically we give examples of those Bratteli-Vershik systems determined by polynomials over \mathbb{N} , the Euler adic, and the reverse Euler adic.

We begin with those determined by polynomials over \mathbb{N} . Every positive integer polynomial of degree d determines a Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$; we will denote this subfamily of systems by $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ and the diagrams by $(\mathcal{D}_{\mathcal{L}})_{p(x)}$.

Let $a_0, a_1, \dots, a_d \in \mathbb{N}$ and $p(x) = a_0 + a_1x + \dots + a_dx^d$. The *Bratteli diagram associated to $p(x)$* is a Bratteli diagram in $\mathcal{D}_{\mathcal{L}}$ such that for every level n , the number of vertices is $dn + 1$ and the number of edges from (n, k) to $(n + 1, k + j)$ is a_j , with $a_j = 0$ for $j > d$ and $j < 0$.

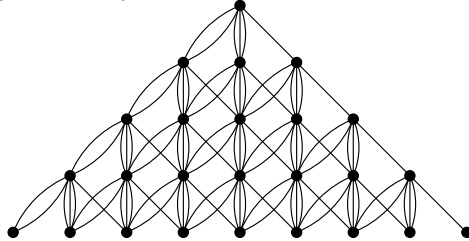


FIGURE 3. The first five levels of $(\mathcal{V}, \mathcal{E})_{2+3x+x^2}$

These diagrams have the property that for any vertex (n, k) the number of paths from the root vertex, $(0, 0)$, into (n, k) is the coefficient of x^k in the polynomial $(p(x))^n$. The most famous example of this is the Pascal adic.

Our next example is the Euler adic, introduced in [?]. The Euler graph, is the Bratteli diagram in $\mathcal{D}_{\mathcal{L}}$ for which the number of vertices at each level n is $n + 1$, and the number of edges connecting vertex (n, k) to vertex $(n + 1, k)$ is $k + 1$ while the number of edges connecting vertex (n, k) to vertex $(n + 1, k + 1)$ is $n - k + 1$.

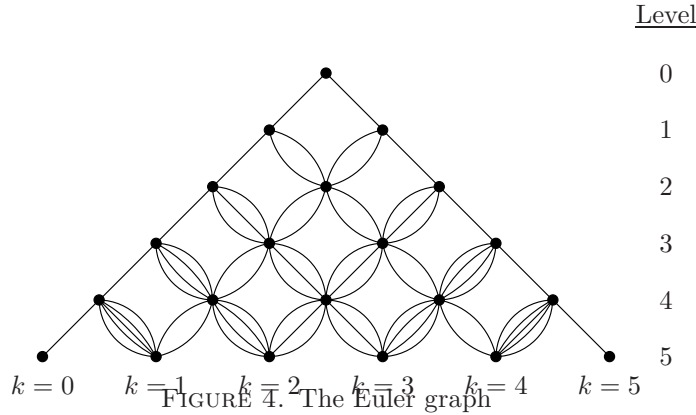


FIGURE 4. The Euler graph

The Euler graph has the property that the number of paths from the root vertex to a vertex (n, k) is the *Eulerian number* $A(n, k)$. That is, the number of permutations $i_1 i_2 \dots i_{n+1}$ of $\{1, 2, \dots, n + 1\}$ with exactly k rises and $n - k$ falls; see [?] for background concerning Eulerian numbers.

The final example is the reverse Euler adic. The reverse Euler graph, is the Bratteli diagram in $\mathcal{D}_{\mathcal{L}}$ for which the number of vertices at each level n is $n + 1$, and the number of edges connecting vertex (n, k) to vertex $(n + 1, k)$ is $n - k + 1$ while the number of edges connecting vertex (n, k) to vertex $(n + 1, k + 1)$ is $k + 1$; the reverse of the connection in the Euler graph.

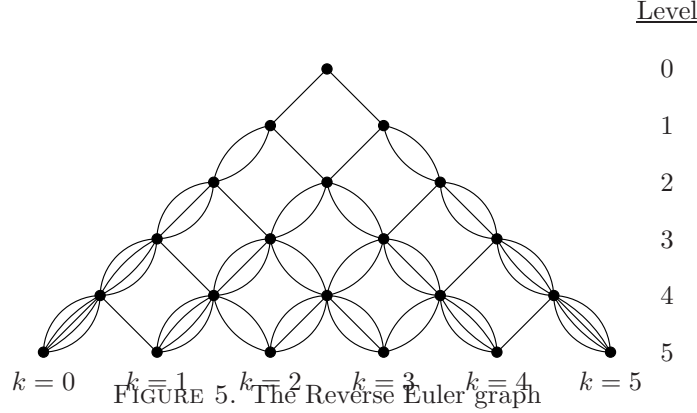


FIGURE 5. The Reverse Euler graph

The reverse Euler graph has the property that the number of paths from the root vertex to a vertex (n, k) is $(n + 1)!$.

3. DIMENSION GROUPS

Every Bratteli diagram can be completely described by a sequence of *incidence* matrices. For any pair of consecutive levels $n - 1$ and n , with v_{n-1} vertices on level $n - 1$ and v_n on level n , the incidence matrix D_n is a $v_n \times v_{n-1}$ matrix such that $[D_n]_{i,j}$ is the number of edges connecting vertices $(n - 1, j)$ and (n, i) .

For every Bratteli diagram $(\mathcal{V}, \mathcal{E})$ there is an associated ordered group called the dimension group and denoted by $K_0(\mathcal{V}, \mathcal{E})$. Explicitly it is the direct limit of the following directed system:

$$\mathbb{Z}^{|V_0|=1} \xrightarrow{\phi_1} \mathbb{Z}^{|V_1|} \xrightarrow{\phi_2} \mathbb{Z}^{|V_2|} \xrightarrow{\phi_3} \dots$$

where for each $i = 1, 2, \dots$ ϕ_i is the group homomorphism determined by the incidence matrix between levels $i - 1$ and i of the Bratteli diagram. The positive set consists of the equivalence classes for which there is a nonnegative vector representative. The equivalence class of $1 \in \mathbb{Z}$ is called the *distinguished order unit*. It is of interest to note that $K_0(\mathcal{V}, \mathcal{E})$ is not dependent on the ordering of the edges or on the associated dynamical system. For further references on ordered groups and dimension groups see [?], [?], [?] and [?].

If (X, T) is any dynamical system let $C(X, \mathbb{Z})$ denote the additive group of continuous functions from the space X to \mathbb{Z} and define

$$\partial_T C(X, \mathbb{Z}) = \{g \circ T - g \mid g \in C(X, \mathbb{Z})\}.$$

The elements of $\partial_T C(X, \mathbb{Z})$ are called the *coboundaries* of (X, ϕ) . In the case that T is a homeomorphism, $K^0(X, T)$ is defined to be $C(X, \mathbb{Z}) / \partial_T C(X, \mathbb{Z})$.

If an ordered Bratteli diagram has X_{\max} and X_{\min} both one point sets, then the diagram is said to be *essentially simple*.

Theorem 3.1 (Herman, Putnam, Skau [?]). *Let $(\mathcal{V}, \mathcal{E}, \geq)$ be an essentially simple ordered Bratteli diagram and let (X, T) be its associated Bratteli-Vershik system. Then there is an order isomorphism*

$$\theta : K_0(\mathcal{V}, \mathcal{E}) \rightarrow K^0(X, \phi)$$

which maps the distinguished order unit of $K_0(\mathcal{V}, \mathcal{E})$ to the equivalence class of the constant function 1.

More discussion of this result can be found in [?, ?].

In the case of systems in $\mathcal{S}_{\mathcal{L}}$, which are not essentially simple, $\partial_T C(X, \mathbb{Z})$ may not be contained in $C(X, \mathbb{Z})$ as T is not continuous everywhere. Nevertheless, by slightly adjusting the definition of $K^0(X, T)$ to be $C(X, \mathbb{Z})/(\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z}))$ we can achieve a result similar to Theorem 3.1.

Theorem 3.2. *For $(X, T) \in \mathcal{S}_{\mathcal{L}}$, there is an order isomorphism*

$$K_0(\mathcal{V}, \mathcal{E}) \cong K^0(X, T)$$

which maps the distinguished order unit of $K_0(\mathcal{V}, \mathcal{E})$ to the equivalence class of the constant function 1.

Before we give the proof, we introduce some useful notation. Let $(\mathcal{V}, \mathcal{E}, \geq)$ be an ordered Bratteli diagram. For $n = 1, 2, \dots$ and $0 \leq k \leq dn$ define $\dim(n, k)$ to be the number of finite paths from the root vertex, $(0, 0)$ into the vertex (n, k) . For any vertex $(n, k) \in \mathcal{V}$ there is a cylinder determined by the path from the root vertex to (n, k) for which all the edges are minimal (maximal). We will call this the *minimal (maximal) cylinder terminating at vertex (n, k)* . Denote by $Y_n(k, 0)$ the minimal cylinder into vertex (n, k) , and let $Y_n(k, i) = T^i(Y_n(k, 0))$ for $i = 0, 1, \dots, \dim(n, k) - 1$. For each $n = 0, 1, 2, \dots$, denote the union of all the minimal cylinders of length n by Y_n , so that

$$Y_n = \bigcup_{0 \leq k \leq |\mathcal{V}_n| - 1} Y_n(k, 0).$$

Proof. This proof is an adaptation of the dynamical proof of Theorem 3.1 given by Glasner and Weiss in [?]. We will first define a group homomorphism $J : C(X, \mathbb{Z}) \rightarrow K_0(\mathcal{V}, \mathcal{E})$. Then we will define a set B and show that it is a subset of $C(X, \mathbb{Z})$. Then we will show $B = \ker(J)$ by first showing $B \subset \ker(J)$ and then $\ker(J) \subset B$. This will induce a one-to-one group homomorphism $\tilde{J} : C(X, \mathbb{Z})/B \rightarrow K_0(\mathcal{V}, \mathcal{E})$. We will then show that \tilde{J} is surjective and in fact an order isomorphism. Lastly we will show $B = \partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z})$.

Let $f \in C(X, \mathbb{Z})$. Since X is compact, f is bounded and hence takes on only finitely many values. Let $\{l_1, \dots, l_j\}$ be the set of these values and let $U_i = f^{-1}\{l_i\}$ for each i . If $i = 1, \dots, j$ and $\gamma \in U_i$, then there is a cylinder set $C_\gamma \subset U_i$ of the form $[c_0 c_1 \dots c_{N_\gamma - 1}]$ which contains γ . From $\{C_\gamma | \gamma \in X\}$ select a finite subcover $\{C_{\gamma^1}, C_{\gamma^2}, \dots, C_{\gamma^r}\}$. Then for some $i \in \{1, 2, \dots, r\}$, C_{γ^i} is of longest length, $N_1(f)$, and f is constant on any cylinder of length $n \geq N_1(f)$. For $n \geq N_1(f)$ define an element $\tilde{f}_n \in \mathbb{Z}^{dn+1}$ by letting, for each $0 \leq k \leq dn$ and any $\gamma \in Y_n(k, 0)$,

$$\tilde{f}_n(k) = f(\gamma) + f(T\gamma) + f(T^2\gamma) + \dots + f(T^{\dim(n, k)-1}\gamma).$$

D_n denotes the adjacency matrix of the edges connecting levels $n - 1$ and n . Then

$$\tilde{f}_{n+1}(i) = \sum_{j=0}^{nd} \tilde{f}_n(j)(D_n)_{i,j} = (\tilde{f}_n D_n)(i).$$

Therefore the sequence \tilde{f}_n defines an element $J(f) \in K_0(\mathcal{V}, \mathcal{E})$. Clearly $J : C(X, \mathbb{Z}) \rightarrow K_0(\mathcal{V}, \mathcal{E})$ is a group homomorphism.

Let $G = \{g \in C(X, \mathbb{Z}) | \exists N_2(g) \text{ such that } n \geq N_2(g) \implies \forall \gamma \in Y_n, g(\gamma) = c\}$. In other words, g takes the same value on all the minimal cylinders into level n . Define $B = \{g \circ T - g | g \in G\}$.

We now show that $B \subset C(X, \mathbb{Z})$. For $f \in B$ with $f = g \circ T - g$, f is continuous on $X \setminus X_{\max}$, we need to check continuity of f on X_{\max} . Let $m \geq \max\{N_1(g), N_2(g)\}$ be such that g is constant on each cylinder of length m and g is also constant on Y_m . For $\gamma_{\max} \in X_{\max}$ and $\xi \in X$, $d(\gamma_{\max}, \xi) < 2^{-m}$ implies that γ_{\max} and ξ are both in the same maximal cylinder terminating at vertex $(m, k_m(\gamma_{\max}))$, and hence $g(\gamma_{\max}) = g(\xi)$. Since $T(\gamma_{\max})$ and $T(\xi)$ are both in Y_m , we have $(g \circ T)(\gamma) = (g \circ T)(\xi)$. Hence $f(\gamma) = f(\xi)$, and so f is continuous.

We will show that $B = \ker(J)$. If $f = g \circ T - g \in B$, $n \geq \max\{N_1(g), N_2(g)\}$, $0 \leq k \leq dn$, and any $\gamma \in Y_n(k, 0)$, then

$$\tilde{f}_n(k) = f(\gamma) + f(T\gamma) + f(T^2\gamma) + \cdots + f(T^{\dim(n,k)-1}\gamma) = g \circ T^{\dim(n,k)}(\gamma) - g(\gamma).$$

Since both γ and $T^{\dim(n,k)}(\gamma) \in Y_n$ and g is constant on Y_n , $\tilde{f}_n(k) = 0$. Therefore $J(f) = 0$, which implies $B \subset \ker(J)$.

Conversely, if $f \in C(X, \mathbb{Z})$ and $J(f) = 0$, there is an $n > N_1(f)$ for which $\tilde{f}_n = 0$. We will define a function $g \in C(X, \mathbb{Z})$ so that $f = g \circ T - g$. Let $g = 0$ on Y_n . For $1 \leq l \leq \dim(n, k)$, choose any $\gamma \in Y_n(k, 0)$ and let $g \equiv f(\gamma) + f(T\gamma) + \cdots + f(T^{l-1}\gamma)$ on $Y_n(k, l)$. Now g is everywhere defined, and clearly $f = g \circ T - g$ on every cylinder terminating at vertex (n, k) except maybe on the maximal cylinder. However, for $\gamma \in Y_n(k, 0)$, $g(T^{\dim(n,k)}\gamma) = 0$ and $\tilde{f}_n(k) = 0$, so we have

$$\begin{aligned} & g(T^{\dim(n,k)}\gamma) - g(T^{\dim(n,k)-1}\gamma) \\ &= -g(T^{\dim(n,k)-1}\gamma) \\ &= -(f(\gamma) + f(T\gamma) + \cdots + f(T^{\dim(n,k)-2}\gamma)) \\ &= -(f(\gamma) + f(T\gamma) + \cdots + f(T^{\dim(n,k)-1}\gamma)) + f(T^{\dim(n,k)-1}\gamma) \\ &= -\tilde{f}_n(k) + f(T^{\dim(n,k)-1}\gamma) \\ &= f(T^{\dim(n,k)-1}\gamma). \end{aligned}$$

Thus $f = g \circ T - g$ also on the maximal cylinder, and hence $f \in B$. Thus $B = \ker(J)$, and J induces an injective group homomorphism $\tilde{J} : C(X, \mathbb{Z})/B \rightarrow K_0(\mathcal{V}, \mathcal{E})$.

We now show that \tilde{J} is onto and an order isomorphism. Given $a \in K_0(\mathcal{V}, \mathcal{E})$, choose an $n \in \mathbb{Z}_+$ so that the equivalence class a has a representative $a_n \in \mathbb{Z}^{dn+1}$. Define f as follows. For $k = 0, 1, \dots, dn$ and $\gamma \in Y_n(k, 0)$, let $f(\gamma) = a_n(k)$ and elsewhere put $f = 0$. Then $\tilde{f}_n(k) = a_n(k)$, so that $J(f) = a$, and thus \tilde{J} is onto. Clearly \tilde{J} takes positive elements to positive elements, and the preceding argument shows that the unique preimage of every positive element under \tilde{J} is a positive element. Thus $C(X, \mathbb{Z})/B$ is order isomorphic to $K_0(\mathcal{V}, \mathcal{E})$ by the map \tilde{J} , which maps the equivalence class of the constant function 1 to the distinguished order unit of $K_0(\mathcal{V}, \mathcal{E})$.

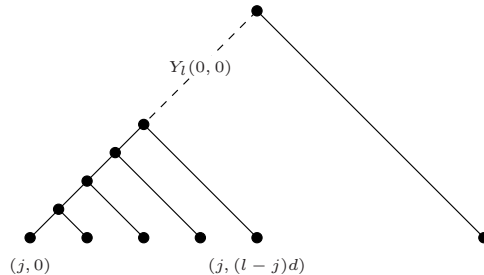


FIGURE 6. Connections from level l to j .

We now show that $\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z}) \subset B$. Let $f \in \partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z})$ be given. Then $f = g \circ T - g$ for some $g \in C(X, \mathbb{Z})$, and f is continuous. We have to

show that there is an $N_2(g)$ so that for each $n \geq N_2(g)$, g takes the same value on all of Y_n . Since $g \in C(X, \mathbb{Z})$, we can choose $l = N_1(g)$ such that g is constant on cylinder sets of length l . Then for every level $j \geq l$, and every $i \in \{0, 1, \dots, (j-l)d\}$, $Y_j(i, 0) \subset Y_l(0, 0)$, (see Figure 6). Now consider $k < dl$ and $\gamma_{\max}^k \in X_{\max}$. Then $T(\gamma_{\max}^k) = \gamma_{\min}^k \in Y_l(ld - k, 0)$.

Since $f = g \circ T - g \in C(X, \mathbb{Z})$, given $\gamma_{\max}^k \in X$ with $k < dl$ there is a $\delta > 0$ such that $d(\gamma_{\max}^k, \xi) < \delta$ implies $f(\gamma_{\max}^k) = f(\xi)$. We will choose a $\xi \in X$ sufficiently close to γ_{\max}^k such that $f(\gamma_{\max}^k) = f(\xi)$ and $g(\gamma_{\max}^k) = g(\xi)$ which implies $g \circ T(\gamma_{\max}^k) = g \circ T(\xi)$. Choose j so that $2^{-j} < \delta$ and $(j-l)d > k+1$. Now let ξ be a path in X such that $\xi_i = (\gamma_{\max}^k)_i$ for each $i = 0, 1, \dots, j-1$ and $\xi_j \neq (\gamma_{\max}^k)_j$. Then $d(\gamma_{\max}^k, \xi) < \delta$, so that $f(\gamma_{\max}^k) = f(\xi)$. Since $j > l$, γ_{\max}^k and ξ are in the same maximal cylinder which terminates at (l, k) , which implies $g(\gamma_{\max}^k) = g(\xi)$. Thus $f(\gamma_{\max}^k) = f(\xi)$ implies $(g \circ T)(\gamma_{\max}^k) = (g \circ T)(\xi)$. Since $s(\xi_j) = (j, k)$, and ξ_j is the first non-maximal edge of ξ , $T\xi$ is in either $Y_j(k, 0)$ or $Y_j(k+1, 0)$ depending on the source of the successor of ξ_j . Since $k+1 < (j-l)d$, we have that $T\xi \in Y_l(0, 0)$. Then $(g \circ T)(\gamma_{\max}^k) = (g \circ T)(\xi)$, and g constant on each cylinder of length l implies $g(Y_l(ld - k), 0) = g(Y_l(0, 0))$. Since $k < dl$ was arbitrary, we have shown that g is constant on all $Y_l(k, 0)$ for $k < dl$. It remains only to show that g takes this same value on $Y_l(dl, 0)$. Consider γ_{\max}^∞ , and choose $j \geq l$ so that $2^{-j} < \delta$. Then $d(\gamma_{\max}^\infty, \gamma_{\max}^{jd}) < \delta$, which implies γ_{\max}^∞ and γ_{\max}^{jd} are both in the maximal cylinder terminating at vertex (l, dl) . Thus $g(\gamma_{\max}^\infty) = g(\gamma_{\max}^{jd})$. Then $f(\gamma_{\max}^\infty) = f(\gamma_{\max}^{jd})$ and $g(\gamma_{\max}^\infty) = g(\gamma_{\max}^{jd})$ implies $(g \circ T)(\gamma_{\max}^\infty) = (g \circ T)(\gamma_{\max}^{jd})$. Thus $T\gamma_{\max}^\infty \in Y_l(dl, 0)$, $T\gamma_{\max}^{jd} \in Y_l(0, 0)$ and g constant on each cylinder of length l implies $g(Y_l(0, 0)) = g(Y_l(dl, 0))$. Hence g is constant on Y_l , as required. \square

We say that a Bratteli diagram is *stationary* if for $n, m \geq 2$, the incidence matrices D_n and D_m are equal. In this case there is a natural method of computing the dimension group. See [?] for the exact construction. In the case of $(\mathcal{D}_{\mathcal{L}})_{p(x)}$ the Bratteli diagrams are clearly not stationary but in a certain sense the group homomorphisms that define the associated dimension groups are stationary and we have the following theorem.

Theorem 3.3. *The dimension group $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ associated to $(\mathcal{V}, \mathcal{E})_{p(x)}$ is order isomorphic to the ordered group $G_{p(x)}$ of rational functions of the form*

$$\frac{r(x)}{p(x)^m},$$

where $r(x)$ is any polynomial with integer coefficients such that $\deg(r(x)) \leq md$. Addition of two elements is given by

$$\frac{r(x)}{p(x)^m} + \frac{s(x)}{p(x)^l} = \frac{r(x) + s(x)p(x)^{m-l}}{p(x)^m}$$

if $l \leq m$. The positive set $(G_{p(x)})_+$ consists of the elements of $G_{p(x)}$ such that there is an l for which the numerator of

$$\frac{r(x)(p(x))^l}{p(x)^{l+m}}$$

has all positive coefficients. The distinguished order unit of $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ is the constant polynomial 1.

Proof. We will construct an order isomorphism from $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ into G . The transposes of the incidence matrices will be used for typographical reasons in the computation in order to make the computations on row vectors. For $p(x) = a_0 + \dots + a_d x^d$, the transpose of the k 'th incidence matrices associated to $(\mathcal{V}, \mathcal{E})_{p(x)}$, ϕ_k will be a $((k-1)d+1) \times (kd+1)$ matrix with

$$(\phi_k)_{ij} = \begin{cases} a_{(j-i)} & \text{if } 0 \leq j-i \leq d \\ 0 & \text{otherwise} \end{cases}$$

For $l \leq m$, define $\phi_{lm} : \mathbb{Z}^{d(l-1)+1} \rightarrow \mathbb{Z}^{dm+1}$ by $\phi_l \phi_{l+1} \dots \phi_m$. We will identify \mathbb{Z}^i with the additive group of polynomials of degree at most $i-1$, $\mathbb{Z}_{i-1}[x]$ in the following manner. For $v = [v_0 \ v_1 \dots v_{i-1}] \in \mathbb{Z}^i$, define $v(x) \in \mathbb{Z}_{i-1}[x]$ by $v(x) = \sum_{j=0}^{i-1} v_j x^j$. Now if $v \in \mathbb{Z}^{dm+1}$, we have $(v\phi_m)(x) = v(x)p(x)$. Under the above correspondence, ϕ_l becomes multiplication by $p(x)$ for all l , and ϕ_{lm} becomes multiplication by $(p(x))^{m-l}$.

Define $\rho_m : \mathbb{Z}_{md}[x] \rightarrow G$ by $\rho_m(r(x)) = \frac{r(x)}{(p(x))^m}$. In order to satisfy the hypothesis of the universal mapping property of direct limits, it needs to be shown that for $l \leq m$, $\rho_l = \rho_m \circ \phi_{lm}$:

$$\begin{aligned} \rho_m \circ \phi_{lm}(r(x)) &= \rho_m(r(x)(p(x))^{m-l}) \\ &= \frac{r(x)(p(x))^{m-l}}{(p(x))^m} \\ &= \frac{r(x)}{(p(x))^l} \\ &= \rho_l(r(x)). \end{aligned}$$

Hence the hypothesis for the universal mapping property of direct limits is satisfied, and the ρ_l are constant on equivalence classes. It follows that there is a unique homomorphism $\rho : K_0(\mathcal{V}, \mathcal{E})_{p(x)} \rightarrow G_{p(x)}$, which can be defined on an equivalence class by taking any representative in \mathbb{Z}^{di+1} and applying ρ_i to it. This is well defined because ρ_i is constant on equivalence classes, and there is only one element of each equivalence class in each \mathbb{Z}^{id+1} . We claim that ρ is an isomorphism.

First we show that ρ is a homomorphism. For without loss of generality, assume $l \leq m$, $r(x) \in \mathbb{Z}_{md}[x]$, and $s(x) \in \mathbb{Z}_{ld}[x]$. Then

$$\begin{aligned} \rho(\overline{r(x) + s(x)}) &= \rho(\overline{r(x) + (p(x))^{m-l}s(x)}) \\ &= \frac{r(x) + (p(x))^{m-l}s(x)}{(p(x))^m} \\ &= \frac{r(x)}{(p(x))^m} + \frac{s(x)}{(p(x))^l} \\ &= \rho(r(x)) + \rho(s(x)). \end{aligned}$$

Now we show that ρ is onto. Given $\frac{r(x)}{(p(x))^m} \in G_{p(x)}$, then for $r(x) \in \mathbb{Z}_{md}[x]$, $\rho(\overline{r(x)}) = \frac{r(x)}{(p(x))^m}$.

Lastly we show that ρ is injective. If $r(x) \in \mathbb{Z}_{md}[x]$, $\rho(\overline{r(x)}) = 0$, then

$$\frac{r(x)}{(p(x))^m} = 0 \text{ therefore } r(x) = 0 \text{ and } \overline{r(x)} = \bar{0}.$$

Hence ρ is an isomorphism, and $G_{p(x)}$ is isomorphic to $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$. In addition, $G_{p(x)}$ is order isomorphic to $K_0(\mathcal{V}, \mathcal{E})_{p(x)}$ because

$$(G_{p(x)})_+ = \left\{ \frac{r(x)}{(p(x))^m} \mid r(x)(p(x))^l \text{ has all positive coefficients for some } l \geq 0 \right\}$$

is exactly the image of the positive set of $\varinjlim \mathbb{Z}^{dk+1}$ under ρ . Finally, the image of 1 is $\frac{1}{(p(x))^0} = 1$. \square

4. SOME ERGODIC ADIC INVARIANT MEASURES

Determining the ergodic adic invariant measures for systems in $\mathcal{S}_{\mathcal{L}}$ depends heavily on the particular system. For instance the Euler adic has a unique fully supported (every cylinder set is given positive measure) invariant measure, [?, ?] while the reverse Euler, [?], and the polynomial systems have one-parameter families of ergodic adic invariant measures. We will devote the remainder of this section to showing that each of the adics given by a positive integer polynomial has a one-parameter family of ergodic adic invariant measure.

For a cylinder set C in $X_{p(x)}$, and any path $\gamma \in X_{p(x)}$, $\dim(C, (n, k_n(\gamma)))$ is the number of paths from the terminal vertex (m, l) of C , to the vertex $(n, k_n(\gamma))$. Define a function $\text{coeff}_{p(x)} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ by $\text{coeff}_{p(x)}(n, k) =$ the coefficient of x^k in the polynomial $(p(x))^n$. Because of the self-similarity of this class of Bratteli diagrams, if C terminates at vertex (m, l) ,

$$\begin{aligned} \dim(C, (n, k_n(\gamma))) &= \text{coeff}_{p(x)}(n - m, k_n(\gamma) - l) \text{ and} \\ \dim(n, k) &= \text{coeff}_{p(x)}(n, k). \end{aligned}$$

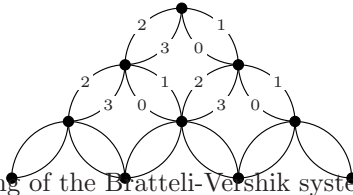
For $n = 0, 1, \dots$ and $d \leq k \leq d(n - 1)$, the number of edges into vertex (n, k) is exactly $a_0 + a_1 + \dots + a_d$. In addition, for every vertex (n, k) the number of edges leaving (n, k) is exactly $a_0 + \dots + a_d$. Because of this it is convenient to use an alphabet to label the edges of paths in $X_{p(x)}$. The alphabet associated to $X_{p(x)}$ will be $A = \{0, 1, \dots, a_0 + a_1 + \dots + a_d - 1\}$. If an edge e is the j 'th edge between vertex (n, k) and $(n + 1, k + i)$ label it

$$\left(\sum_{m=0}^{d-i-1} \right) + (j - 1);$$

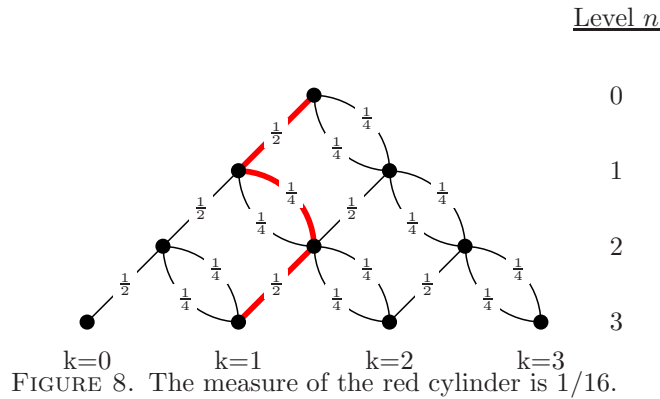
see Figure 7. By labeling in this manner, the lexicographic ordering on comparable edges is consistent with the edge ordering given for the general family $\mathcal{S}_{\mathcal{L}}$. Then any path in $X_{p(x)}$ is uniquely determined by the labeling of its edges, and because of this, we use both X and a one sided infinite sequence in $A^{\mathbb{N}}$ to denote the infinite edge paths on a Bratteli diagram. For ease of notation we will refer to a path by the infinite labeling of its edges, and when the context is clear, we will refer to an edge by its label.

Consider some cylinder set $C = [c_0 c_1 \dots c_{n-1}] \in X_{p(x)}$ and any $T_{p(x)}$ -invariant Borel probability measure μ on $X_{p(x)}$. Define the *weight* w_{c_0} on the edge c_0 to be $\mu([c_0])$. For $n > 0$ and $\mu([c_0 c_1 \dots c_{n-1}]) = 0$ define the weight w_{c_n} on c_n to be 0. For $n > 0$ and $\mu([c_0 \dots c_{n-1}]) > 0$ define w_{c_n} on c_n to be $\mu([c_0 \dots c_n]) / \mu([c_0 \dots c_{n-1}])$. Then $\mu([c_0 \dots c_n]) = w_{c_0} \dots w_{c_n}$.

These weights are well defined because as we will see in Lemma 4.3, all cylinders with the same terminal vertex have the same measure.



Remark 1. In this section we discuss measures, $T_{p(x)}$ -invariant Borel probability measures for which edges with the same label have the same weight. Then for the probability space $(X_{p(x)}, \mathcal{B}, \mu)$ there are at most $a_0 + a_1 + \dots + a_d$ different weights. For each $j \in A$ we will denote by w_j the weight associated to each edge labeled j . Since $(X_{p(x)}, \mathcal{B}, \mu)$ is a probability space, $\sum_{i=0}^{a_0+\dots+a_d-1} w_i = 1$. In view of the labeling of paths by the alphabet A , these measures are Bernoulli and we denote such a measure by $\mathcal{B}(w_{a_0+\dots+a_d-1}, \dots, w_0)$.



In [?, ?], Méla showed that when all the coefficients of $p(x)$ are 1, the invariant ergodic probability measures for $T_{p(x)}$ are the Bernoulli measure $\mathcal{B}(0, \dots, 0, 1)$ and the one-parameter family $\mathcal{B}(q, t_q, \frac{t_q^2}{q}, \frac{t_q^3}{q^2}, \dots, \frac{t_q^n}{q^{n-1}})$, where t_q is the unique solution in $[0, 1]$ to the equation

$$q^n - q^{n-1} + q^{n-1}t + q^{n-2}t^2 + \dots + qt^{n-1} + t^n = 0.$$

Using similar techniques we have extended this result to the following:

Theorem 4.1. *Let $p(x) = a_0 + \dots + a_d x^d$ and let $(X_{p(x)}, T_{p(x)})$ be the Bratteli-Vershik system in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ determined by $p(x)$. If $q \in \left(0, \frac{1}{a_0}\right)$, and t_q is the unique solution in $[0, 1]$ to the equation*

$$(4.1) \quad a_0 q^d + a_1 q^{d-1} t + \dots + a_d t^d - q^{d-1} = 0,$$

then the invariant, fully supported, ergodic probability measures for the adic transformation $T_{p(x)}$ are the one-parameter family of Bernoulli measures

$$\mathcal{B} \left(\underbrace{q, \dots, q}_{a_0 \text{ times}}, \underbrace{t_q, \dots, t_q}_{a_1 \text{ times}}, \underbrace{\frac{t_q^2}{q}, \dots, \frac{t_q^2}{q}}_{a_2 \text{ times}}, \dots, \underbrace{\frac{t_q^n}{q^{n-1}}, \dots, \frac{t_q^n}{q^{n-1}}}_{a_n \text{ times}} \right).$$

Proposition 4.2. *Let $p(x) = a_0 + \dots + a_d x^d$ and $(X_{p(x)}, T_{p(x)})$ be the Bratteli-Vershik system determined by $p(x)$. The only $T_{p(x)}$ invariant, ergodic probability measures that are not fully supported are the Bernoulli measures*

$$\mathcal{B} \left(\underbrace{\frac{1}{a_0}, \dots, \frac{1}{a_0}}_{a_0 \text{ times}}, 0, \dots, 0 \right) \text{ and } \mathcal{B} \left(0, \dots, 0, \underbrace{\frac{1}{a_n}, \dots, \frac{1}{a_n}}_{a_n \text{ times}} \right).$$

The proofs of Theorem 4.1 and Proposition 4.2 use many other results and definitions, which are presented below. Proposition 4.7 shows that every invariant fully supported ergodic probability measure for $(X_{p(x)}, T_{p(x)})$ must be Bernoulli. Proposition 4.8 says that the Bernoulli measures that are $T_{p(x)}$ -invariant are in fact ergodic. Proposition 4.9 shows which Bernoulli measures are $T_{p(x)}$ -invariant. This will prove Theorem 4.1. We will then conclude with the proof of Proposition 4.2.

Lemma 4.3. *Any non-atomic measure on $X_{p(x)}$ is $T_{p(x)}$ -invariant if and only if all cylinders with the same terminal vertex have the same measure.*

Proof. (\Rightarrow) Let $(n, k) \neq (0, 0)$ be a vertex of $(\mathcal{V}, \mathcal{E})_{p(x)}$. Consider the maximal path from $(0, 0)$ to (n, k) and the cylinder set, C_{\max} , defined by this path. There exists an i such that $T_{p(x)}^{-i}(C_{\max})$ is the minimal cylinder $Y_n(k, 0)$ determined by the minimal path from $(0, 0)$ to (n, k) . Since the measure is $T_{p(x)}$ -invariant, the elements of the set $\{T_{p(x)}^{-j}(C_{\max})\}_{j=0}^i$ all have the same measure. Because all the cylinders with terminal vertex (n, k) are contained in the above set, all cylinders with terminal vertex (n, k) have the same measure.

(\Leftarrow) It is enough to show that for each cylinder set C , $T_{p(x)}^{-1}C$ has the same measure as C . Let C be any cylinder set, with terminal vertex (n, k) . Suppose that C is not minimal. Since C is not minimal, $T_{p(x)}^{-1}(C)$ has terminal vertex (n, k) , and hence the same measure as C .

If C is minimal, it can be decomposed into a disjoint union of minimal cylinders and at most a countable number of infinite paths. since the measure is non-atomic, the measure of C equals the measure of $T^{-1}C$. \square

The following is a lemma of Vershik that is proved using the ergodic theorem and the indicator function for the cylinder set C .

Lemma 4.4 (Vershik [?, ?]). *If μ is an invariant non-atomic ergodic probability measure for the adic transformation $T_{p(x)}$, then for every cylinder set C ,*

$$\mu(C) = \lim_{n \rightarrow \infty} \frac{\dim(C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))} \text{ for } \mu\text{-a.e. } \gamma \in X$$

The following lemma shows that the invariant and ergodic probability measures for $T_{p(x)}$ are also invariant for the one-sided shift σ on $A^{\mathbb{N}}$.

Lemma 4.5. *For each $\gamma \in X_{p(x)}$ and $j \in A$, let $\sigma_j \gamma = j\gamma_0\gamma_1\dots$. If μ is invariant and ergodic for $T_{p(x)}$, then for any cylinder set C ,* $\mu(C) = \mu(\sigma_0 C) + \mu(\sigma_1 C) + \dots + \mu(\sigma_{a_d+\dots+a_0-1} C) =$

Proof. Define a function $g : A \rightarrow \{0, 1, \dots, d\}$ such that if the letter $j \in A$ is the label of an edge which connects vertex (n, k) to $(n, k + i)$, then $g(j) = i$. Then for any cylinder set $C = [c_0\dots c_{n-1}]$ which terminates at vertex (n, k) , we have

$$\sum_{i=0}^{n-1} g(c_i) = k.$$

Let C be a cylinder set with terminal vertex (m, l) . For almost every γ in X and each $j \in A$ we have

$$\mu(\sigma_j C) = \lim_{n \rightarrow \infty} \frac{\dim(\sigma_j C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))}.$$

Denote by C^j the cylinder set extended by j . The terminal vertex of C^j is $(m + 1, l + g(j))$, and since the first $m + 1$ edges of $\sigma_j C$ are a permutation of those of C^j , the terminal vertex of $\sigma_j C$ is also $(m + 1, l + g(j))$. Hence for all $n > m$, $\dim(C^j, (n, k_n(\gamma))) = \dim(\sigma_j C, (n, k_n(\gamma)))$.

The set of finite paths starting from (m, l) and ending at $(n, k_n(\gamma))$ can be divided into $a_d + \dots + a_0$ groups, according to whether the edge is labeled $0, 1, \dots, \left(\sum_{i=0}^d a_i\right) - 1$.

Then we have, $\dim(C, (n, k_n(\gamma)))$

$$\begin{aligned} &= \dim(C^0, (n, k_n(\gamma))) + \dots + \dim(C^{(\sum_{i=0}^d a_i)-1}, (n, k_n(\gamma))) \\ &= \dim(\sigma_0 C, (n, k_n(\gamma))) + \dots + \dim(\sigma_{(\sum_{i=0}^d a_i)-1} C, (n, k_n(\gamma))). \end{aligned}$$

Therefore, $\frac{\dim(C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))}$

$$= \frac{\dim(\sigma_0 C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))} + \dots + \frac{\dim(\sigma_{(\sum_{i=0}^d a_i)-1} C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))}.$$

Taking limits as $n \rightarrow \infty$, $\mu(C) = \mu(\sigma_0 C) + \dots + \mu(\sigma_{a_d+\dots+a_0-1} C)$. \square

Lemma 4.6. *For $j_0, j_1 \in A$, $\dim(C^{j_0}, (n, k_n(\gamma))) = \dim(C^{j_1 j_0}, (n, k_{n+1}(\sigma_{j_1} \gamma)))$.*

Proof. Assume that C terminates at vertex (m, k) . Then C^{j_0} terminates at $(m + 1, l + g(j_0))$, where $g(j)$ is as in the proof of Lemma 4.5. Hence, $\dim(C^{j_0}, (n, k_n(\gamma))) = \text{coeff}_{p(x)}(n - (m + 1), k_n(\gamma) - (l + g(j_0)))$. Also, $k_{n+1}(\sigma_{j_1} \gamma) = k_n(\gamma) + g(j_1)$, and $C^{j_1 j_0}$ terminates at vertex $(m + 2, l + g(j_1) + g(j_0))$. Hence

$$\begin{aligned} \dim(C^{j_1 j_0}, (n, k_{n+1}(\sigma_{j_1} \gamma))) &= \text{coeff}_{p(x)}(n + 1 - (m + 2), k_n(\gamma) + g(j_1) - (l + g(j_1) + g(j_0))) \\ &= \text{coeff}_{p(x)}(n - (m + 1), k_n(\gamma) - (l + g(j_0))) \\ &= \dim(C^{j_0}, (n, k_n(\gamma))). \end{aligned}$$

\square

Proposition 4.7. *Every $T_{p(x)}$ -invariant fully supported ergodic probability measure for $(X_{p(x)}, T_{p(x)})$ is Bernoulli.*

Proof. Let μ be a $T_{p(x)}$ -invariant fully supported ergodic probability measure for $(X_{p(x)}, T_{p(x)})$. To prove that μ is a Bernoulli measure, it is enough to show that for each $i \in A$ there exists a number p_i such that for every cylinder set C , $\frac{\mu(C^i)}{\mu(C)} = p_i$.

Now for any $\gamma \in X_{p(x)}$,

$$\frac{\dim(C^i, (n, k_n(\gamma)))}{\dim(C, (n, k_n(\gamma)))} = \frac{\dim(C^{ji}, (n, k_{n+1}(\sigma_j \gamma)))}{\dim(C^j, (n, k_{n+1}(\sigma_j \gamma)))}.$$

By Lemma 4.4, there exists a set E of full measure such that for all $\gamma \in E$ and all $i \in A$

$$\frac{\mu(C^i)}{\mu(C)} = \lim_{n \rightarrow \infty} \frac{\dim(C^i, (n, k_n(\gamma)))}{\dim(C, (n, k_n(\gamma)))}.$$

If there is $j \in A$ such that $E \cap \sigma_j E = \emptyset$, then $\mu(E \cap \sigma_j E) = 0$, and, since E has full measure, $\mu(\sigma_j E) = 0$. Denote by $[r]$ the cylinder set $\{\gamma \in X : \gamma_0 = r\}$.

Then by Lemma 4.5:

$$1 = \mu(E) = \sum_{r \neq j} \mu(\sigma_r E) \leq \sum_{r \neq j} \mu[r] \leq 1 \text{ which implies } \mu([j]) = 0,$$

contradicting our earlier assumption that μ has full support. Hence there exists $\gamma \in E \cap \sigma_j E$. Let ξ be the path in E such that $\sigma_j \xi = \gamma$; then

$$\lim_{n \rightarrow \infty} \frac{\dim(C^i, (n, k_n(\xi)))}{\dim(C, (n, k_n(\xi)))} = \lim_{n \rightarrow \infty} \frac{\dim(C^{ji}, (n, k_{n+1}(\sigma_j \xi)))}{\dim(C^j, (n, k_{n+1}(\sigma_j \xi)))},$$

$$\text{showing that } \frac{\mu(C^i)}{\mu(C)} = \frac{\mu(C^{ji})}{\mu(C^j)}.$$

Then for any cylinder set $C = [c_0 c_1 \dots c_{m-1}]$ we have:

$$\frac{\mu(C^i)}{\mu(C)} = \frac{\mu([c_0 \dots c_{m-2}]^{c_{m-1} i})}{\mu([c_0 \dots c_{m-2}]^{c_{m-1}})} = \frac{\mu([c_0 \dots c_{m-2}]^i)}{\mu([c_0 \dots c_{m-2}])} = \dots = \frac{\mu([c_0]^i)}{\mu([c_0])}.$$

Also, for all $j, k, l \in A$ we have:

$\mu([jli]) = \mu([lji])$, since $[jli]$ and $[lji]$ have the same terminal vertex. Then

$$\frac{\mu([j]^{li})}{\mu([j]^l)} = \frac{\mu([l]^{ji})}{\mu([l]^j)} \text{ so that } \frac{\mu([j]^i)}{\mu([j])} = \frac{\mu([l]^i)}{\mu([l])}.$$

This shows that $\frac{\mu(C^i)}{\mu(C)}$ is independent of C , and hence equal to $\mu([i])$. Therefore μ is a Bernoulli. \square

Proposition 4.8. *The $T_{p(x)}$ -invariant Bernoulli measures on $X_{p(x)}$ are ergodic.*

Proof. Define the random variable Z_i on $X_{p(x)}$ by letting $Z_i(\gamma)$ be the label on the $i-1$ 'th edge of γ . Since the probability measure is Bernoulli, the Z_i are independent and identically distributed. If B is a set that depends symmetrically on Z_1, \dots, Z_n , then $\gamma \in B$ implies that $\{\xi \in X | \xi_0 \xi_1 \dots \xi_{n-1} \text{ is a permutation of } \gamma_0 \gamma_1 \dots \gamma_{n-1} \text{ and for } m \geq n, \xi_m = \gamma_m\}$ is also in B . If \mathcal{S}_n is σ -algebra set generated by such B , the Hewitt-Savage theorem implies that $\mathcal{S} = \cap_{n=1}^{\infty} \mathcal{S}_n$ is trivial.

Let \mathcal{T}_n be the σ -algebra generated by sets B' such that if $\gamma \in B'$, then $\{\xi \in X | \text{for } m \geq n, \xi_m = \gamma_m\}$ is also in B' . Then for each generator B' of \mathcal{T}_n , there are a finite number of generators B_i of \mathcal{S}_n such that $\cup_{i=1}^m B_i = B'$. Hence $B' \subset \mathcal{S}_n$. Then $\mathcal{T}_n \subset \mathcal{S}_n$, and $\cap_{i=1}^{\infty} \mathcal{T}_n = \mathcal{T} \subset \mathcal{S}$. Since \mathcal{S} is trivial, so is \mathcal{T} . But \mathcal{T} is the

σ -algebra of $T_{p(x)}$ -invariant sets. Therefore the invariant Bernoulli measures for $T_{p(x)}$ are ergodic. \square

It remains to determine which Bernoulli measures are invariant.

Proposition 4.9. *The Bernoulli measures invariant for the adic transformation $T_{p(x)}$ are the fully supported ones described in Theorem 4.1, along with*

$$\mathcal{B} \left(\underbrace{\frac{1}{a_0}, \dots, \frac{1}{a_0}}_{a_0 \text{ times}}, 0, \dots, 0 \right) \text{ and } \mathcal{B} \left(0, \dots, 0, \underbrace{\frac{1}{a_d}, \dots, \frac{1}{a_d}}_{a_d \text{ times}} \right).$$

Proof. Recall that any edge label j has weight $w(j)$. Recall the definition of $g : A \rightarrow \{0, 1, \dots, d\}$ as given in the proof of Lemma 4.5. By Lemma 4.3, $g(j_1) = g(j_2)$ implies $w(j_1) = w(j_2)$. For $0 \leq t \leq d$, define $p_t = w(j)$ whenever $g(j) = t$. Then

$$(4.2) \quad a_0 p_0 + \dots + a_d p_d = 1.$$

For $s \in \mathbb{N}$ and $i_k, j_k \in \{0, 1, \dots, d\}$ for all $k = 0, 1, \dots, s$, Lemma 4.3 implies that a Bernoulli measure is $T_{p(x)}$ invariant if and only if whenever

$$(4.3) \quad \sum_{k=0}^s i_k = \sum_{k=0}^s j_k, \text{ we have } \prod_{k=0}^s p_{i_k} = \prod_{k=0}^s p_{j_k}.$$

Assume for now that $p_0, p_1 > 0$. Claim: Equation 4.3 is satisfied if and only if $p_0 p_j = p_1 p_{j-1}$ for $1 \leq j \leq d$.

Clearly Equation 4.3 implies $p_0 p_j = p_1 p_{j-1}$. It remains to be shown that $p_0 p_j = p_1 p_{j-1}$ implies Equation 4.3.

For $1 \leq j \leq d$ we will assume

$$(4.4) \quad p_0 p_j = p_1 p_{j-1}.$$

We will use induction to prove to prove our claim. The hypothesis is that for $i_0, i_1, \dots, i_{s-1}, j_0, j_1, \dots, j_{s-1}$ in $\{0, 1, \dots, d\}$, whenever

$$(4.5) \quad i_0 \dots + i_{s-1} = j_0 + \dots + j_{s-1}, \text{ we have } \prod_{k=0}^{s-1} p_{i_k} = \prod_{k=0}^{s-1} p_{j_k}.$$

We will show that for $i_0, i_1, \dots, i_s, j_0, j_1, \dots, j_s$ in $\{0, 1, \dots, d\}$, whenever

$$i_0 \dots + i_s = j_0 + \dots + j_s, \text{ we have } \prod_{k=0}^s p_{i_k} = \prod_{k=0}^s p_{j_k}.$$

We now show the base case. For $1 \leq i \leq d$ and $0 \leq k \leq d-1$, Equation 4.4 implies

$$p_i = \frac{p_1}{p_0} p_{i-1} \text{ and } p_k = \frac{p_0}{p_1} p_{k+1}.$$

Hence

$$p_i p_k = \frac{p_1}{p_0} p_{i-1} \frac{p_0}{p_1} p_{k+1} = p_{i-1} p_{k+1}.$$

For $i, k, l, m \in \{0, 1, \dots, d\}$ we then have that whenever $i + k = l + m$, $p_i p_k = p_l p_m$, hence we have shown the base case.

Now consider $i_0, i_1, \dots, i_s, j_0, j_1, \dots, j_s$ in $\{0, 1, \dots, d\}$ such that

$$i_0 + \dots + i_s = j_0 + \dots + j_s.$$

Then $i_0 + \cdots + i_s - i_s - j_s = j_0 + \cdots + j_s - j_s - i_s$, hence

$$i_0 + \cdots + i_{s-1} - j_s = j_0 + \cdots + j_{s-1} - i_s.$$

There also exist l_0, l_1, \dots, l_{s-2} in $\{0, 1, \dots, d\}$ such that

$$l_0 + \cdots + l_{s-2} = i_0 + \cdots + i_{s-1} - j_s = j_0 + \cdots + j_{s-1} - i_s.$$

Adding j_s to both sides, we see that $l_0 + \cdots + l_{s-2} + j_s = i_0 + \cdots + i_{s-1}$, hence the induction hypothesis implies

$$(4.6) \quad p_{j_s} \prod_{k=0}^{s-2} p_{l_k} = \prod_{k=0}^{s-1} p_{i_k}.$$

Likewise, $l_0 + \cdots + l_{s-2} + i_s = j_0 + \cdots + j_{s-1}$, and the induction hypothesis implies

$$(4.7) \quad p_{i_s} \prod_{k=0}^{s-2} p_{l_k} = \prod_{k=0}^{s-1} p_{j_k}.$$

Combining Equation 4.6 and Equation 4.7 we see that

$$\prod_{k=0}^s p_{i_k} = p_{i_s} p_{j_s} \prod_{k=0}^{s-2} p_{l_k} = p_{j_s} p_{i_s} \prod_{k=0}^{s-2} p_{l_k} = \prod_{k=0}^s p_{j_k}.$$

Therefore we have proved the claim and the Bernoulli measures are $T_{p(x)}$ invariant if and only if for $1 \leq j \leq d$,

$$p_0 p_j = p_1 p_{j-1}.$$

For simplicity of notation define $p_0 = q$ and $p_1 = t$. For $1 \leq j \leq d$, $p_j = \frac{t}{q} p_{j-1}$. Hence every p_j can be defined inductively by t and q . In particular, for $1 \leq j \leq d$,

$$p_j = \frac{t^j}{q^{j-1}}.$$

By Equation 4.2

$$a_0 q + a_1 t + a_2 \frac{t^2}{q} + \cdots + a_d \frac{t^d}{q^{d-1}} = 1.$$

Multiplying through by q^{d-1} and simplifying, we see that

$$(4.8) \quad a_0 q^d + a_1 q^{d-1} t + \cdots + a_d t^d - q^{d-1} = 0.$$

To conclude that p_1, \dots, p_d are completely determined by the choice of $p_0 = q$, it remains only to show that for each $q \in (0, 1/a_0)$, Equation 4.8 has a unique solution in $[0, 1]$.

Consider $m(t) = a_0 q^d + a_1 q^{d-1} t + \cdots + a_d t^d - q^{d-1}$. Then $m(0) = a_0 q^d - q^{d-1} = q^{d-1}(a_0 q - 1) \leq 0$, since $a_0 q \leq 1$. Also, $m(1) = a_0 q^d + a_1 q^{d-1} + \cdots + a_d - q^{d-1} > 0$, since $a_1 \geq 1$ implies $a_1 q^{d-1} - q^{d-1} \geq 0$. By the intermediate value theorem, there exists a root in $[0, 1]$. Now, $m'(t) = a_1 q^{d-1} + \cdots + d a_d t^{d-1}$ is strictly greater than 0 on $[0, 1]$, so that $m(t)$ is strictly increasing on $[0, 1]$; therefore there is a unique solution t_q to $m(t) = 0$ in the interval $[0, 1]$.

If $p_0 = 1/a_0$, $a_0 p_0 = 1$ and all other $p_i = 0$, hence the $T_{p(x)}$ -invariant measure μ is supported on the paths for which $k_n(\gamma) = 0$ for all $n \geq 0$. Finally, if $i \leq d-2$ and $p_i = 0$, then $i+2 \leq d$, and $p_i p_{i+2} = p_{i+1} p_{i+1} = 0$. Hence $p_{i+1} = 0$. If $p_0 = 0$ then $p_i = 0$ for all $0 \leq i < d$. This implies that the only nonzero probability is p_d and $a_d p_d = 1$, hence $p_d = 1/a_d$ and the $T_{p(x)}$ -invariant measure μ is supported on the set of paths for which $k_n(\gamma) = dn - 1$ for all $n \geq 0$.

□

We now have enough tools to prove Theorems 4.1 and 4.2.

Proof of Theorem 4.1. Direct from Proposition 4.9, Proposition 4.8, and Proposition 4.7. □

Proof of Proposition 4.2. According to Proposition 4.9 and Proposition 4.8, the Bernoulli measures

$$\mathcal{B} \left(\underbrace{\frac{1}{a_0}, \dots, \frac{1}{a_0}}_{a_0 \text{ times}}, 0, \dots, 0 \right) \text{ and } \mathcal{B} \left(0, \dots, 0, \underbrace{\frac{1}{a_n}, \dots, \frac{1}{a_n}}_{a_n \text{ times}} \right)$$

are ergodic and invariant for $(X_{p(x)}, T_{p(x)})$. It remains to show that these are the only invariant, ergodic probability measures that are not fully supported on $X_{p(x)}$.

By Proposition 1.1, the only proper closed invariant sets are those for which the tails are eventually diagonal.

Define A_k to be the closed invariant set

$$A_k = \{\gamma \in X \mid \text{either } k_n(\gamma) \leq k \text{ for all } n \text{ or } dn - k_n(\gamma) \leq k \text{ for all } n\}.$$

Define C to be the maximal cylinder from the root vertex to vertex (l, k) , where l is the first level for which $dl - k > k$. In other words, C is the maximal cylinder of shortest length such that for each $\gamma \in C$ and $n \geq l$, $k_n(\gamma) = k$.

If μ is a $T_{p(x)}$ -invariant ergodic probability measure that is not fully supported on $X_{p(x)}$, we will compute $\mu(C)$ using Lemma 4.4. If there is a set of positive measure $B_k \subset A_k$ for which every $\gamma \in B_k$ and $n \geq l$ has $k_n(\gamma) \neq k$, then for all $n \geq l$, $\dim(C, (n, k_n(\gamma))) = 0$, hence $\mu(C) = 0$. Therefore we will assume that almost every γ in A_k has $k_n(\gamma) \equiv k$ for sufficiently large n . This implies that the measure μ is supported on the paths for which $k_n(\gamma) \leq k$.

Then

$$\mu(C) = \lim_{n \rightarrow \infty} \frac{\dim(C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))}.$$

It is clear that $\dim(C, (n, k_n(\gamma))) = a_0^{n-l}$. We will now find a lower bound for $\dim(n, k_n(\gamma))$. Let $m \in \mathbb{Z}_+$ such that $k - md > 0$ and $k - (m+1)d \leq 0$. Then $dm + i = k$ for some $i \in \{1, 2, \dots, d\}$. Recall the function $g(j)$ defined in the proof of Lemma 4.5. We will only count the paths $\xi \in A_k$ for which $g(\xi_0) = g(\xi_1) = \dots = g(\xi_{j-1}) = 0$, $g(\xi_j) = i$, $g(\xi_{j+1}) = g(\xi_{j+2}) = \dots = g(\xi_{j+m}) = d$, and $g(\xi_{j+m+1}) = g(\xi_{j+m+2}) = \dots = g(\xi_{n-1}) = 0$, where $0 \leq j \leq n - (m+1)$ (see Figure 9). The range of ξ_{n-1} is $(n, k_n(\gamma))$. These paths form a subset of the paths which are counted when computing $\dim(n, k_n(\gamma))$. For a fixed j , the number of such paths is

$$a_0^j a_i a_d^m a_0^{n-m-j-1} = a_0^{n-m-1} a_i a_d^m.$$

Letting j range over 0 to $n - (m+1)$ we see that

$$\dim(n, k_n(\gamma)) \geq (n - m) a_0^{n-m-1} a_i a_d^m. \text{ Hence}$$

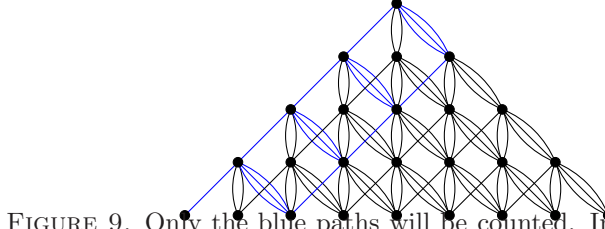


FIGURE 9. Only the blue paths will be counted. In this case $i = d = 2$ and $m = 0$.

$$\begin{aligned} \mu(C) &= \lim_{n \rightarrow \infty} \frac{\dim(C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))} \\ &\leq \lim_{n \rightarrow \infty} \frac{a_0^{n-l}}{(n-m)a_0^{n-m-1}a_i a_d^m} = \lim_{n \rightarrow \infty} \frac{a_0^{m+1-l}}{(n-m)a_i a_d^m} \\ &= 0. \end{aligned}$$

By the invariance of $T_{p(x)}$, no cylinder whose terminal vertex is (j, k) , where $j \geq l$, has positive measure. Using this same argument for vertices (j, i) where $i \leq k$, we see that the only edges on which μ is supported are the edges on the far left of the diagram. In this case we have an odometer, and the measure is as stated in the proposition. A symmetric argument shows that the only measure supported on paths for which $dn - k_n(\gamma) \leq k$ for all $n \geq 0$ is supported on the paths for which $k_n(\gamma) = dn$ for all $n \geq 0$. Hence μ is as stated in the proposition. \square

5. EIGENVALUES AND TOTAL ERGODICITY

In this section we discuss the eigenvalues of various systems in $\mathcal{S}_{\mathcal{L}}$. We will show that every Bratteli-Vershik system determined by a positive integer polynomial of degree 1 for which either of the coefficients is greater than 1 has at least one non-trivial root of unity as an eigenvalue. In contrast we will show that the Euler adic has no root of unity (other than one) as an eigenvalue. In [?] it is shown that the reverse Euler adic has every root of unity as an eigenvalue.

Theorem 5.1. *Let $(X_{p(x)}, T_{p(x)})$ be the Bratteli-Vershik system in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ determined by $p(x) = a_0 + a_1x$, with fully-supported, $T_{p(x)}$ -invariant, ergodic probability measure μ . Then $e^{2\pi i/(a_0 a_1)}$, $e^{2\pi i/a_0}$, and $e^{2\pi i/a_1}$ are eigenvalues of $T_{p(x)}$.*

Proof. The sets $\{\gamma \in X_{p(x)} | k_n(\gamma) = 0 \text{ for all } n = 0, 1, \dots\}$ and $\{\gamma \in X_{p(x)} | k_n(\gamma) = n \text{ for all } n = 0, 1, \dots\}$ are both $T_{p(x)}$ -invariant sets. Since μ is ergodic and has full support these are sets of measure 0. Recall that the minimal cylinder into vertex (n, k) , is denoted $Y_n(k, 0)$ and that $Y_n(k, i) = T^i(Y_n(k, 0))$ for $i = 0, 1, \dots, \dim(n, k) - 1$. For μ -almost every $\gamma \in X_{p(x)}$ there exist $n \geq 0$, $0 < k < n$, and $0 \leq j \leq \dim(n, k) - 1$ for which $\gamma \in Y_n(k, j)$.

Define the function $f : X_{p(x)} \rightarrow \mathbb{C}$ by the following: for $n > 1$, $0 < k < n$, and $0 \leq j < \dim(n, k)$,

$$f(Y_n(k, j)) = (e^{2\pi i/(a_0 a_1)})^{j+1},$$

and $f = 0$ elsewhere.

In order to show that f is well defined it is enough to show that for a minimal cylinder C and an extension C^j of C

$$f(C^j) = e^{2\pi i/(a_0 a_1)}.$$

We will divide the argument into two cases, the case when C is extended by an edge $j \in \{0, 1, \dots, a_1 - 1\}$ (C extended to the right) and the case when C is extended by an edge $j \in \{a_1, a_1 + 1, \dots, a_1 + a_0 - 1\}$ (C extended to the left). First assume that C terminates at vertex $0 < k < n$, and let C^j be the extension of the cylinder C by the edge $j \in \{0, 1, \dots, a_1 - 1\}$, which has terminal vertex $(n+1, k+1)$. Then $C^j = Y_{n+1}(k+1, j \dim(n, k))$. Since $0 < k < n$,

$$(5.1) \quad \dim(n, k) = \binom{n}{k} a_0^{n-k} a_1^k = 0 \pmod{a_0 a_1},$$

and therefore

$$f(C^j) = (e^{2\pi i/(a_0 a_1)})^{j \dim(n, k)+1} = (e^{2\pi i/(a_0 a_1)})^{j \dim(n, k)} e^{2\pi i/(a_0 a_1)} = e^{2\pi i/(a_0 a_1)}.$$

Now let C^j be the extension of the cylinder C by an edge $j \in \{a_1, a_1 + 1, \dots, a_1 + a_0 - 1\}$. Then $C^j = Y_{n+1}(k, a_1 \dim(n, k-1) + (j-a_1) \dim(n, k))$. Since $0 < k < n$,

$$a_1 \dim(n, k-1) = \binom{n}{k-1} a_0^{n-k+1} a_1^k = 0 \pmod{a_0 a_1},$$

and from Equation 5.1 we see that,

$$\begin{aligned} f(C^j) &= (e^{2\pi i/(a_0 a_1)})^{a_1 \dim(n, k-1) + (j-a_1) \dim(n, k)+1} \\ &= (e^{2\pi i/(a_0 a_1)})^{a_1 \dim(n, k-1)} (e^{2\pi i/(a_0 a_1)})^{(j-a_1) \dim(n, k)} e^{2\pi i/(a_0 a_1)} \\ &= e^{2\pi i/(a_0 a_1)}. \end{aligned}$$

Hence f is well defined μ -almost everywhere.

For $n > 1$, $0 < k < n$, $0 \leq j < \dim(n, k) - 1$ and $\gamma \in Y_n(k, j)$, it is clear that $f(T_{p(x)}\gamma) = e^{2\pi i/(a_0 a_1)} f(\gamma)$. For $n \geq 0$, $0 < k < n$, and $\gamma \in Y_n(k, \dim(n, k) - 1)$, there are $m \geq 0$ and $0 < l < m$ such that $T_{p(x)}\gamma \in Y_m(k, 0)$. Then $f(\gamma) = 1$ and $f(T_{p(x)}\gamma) = e^{2\pi i/(a_0 a_1)}$. Hence for μ -almost every $\gamma \in X$, $f(T_{p(x)}\gamma) = e^{2\pi i/(a_0 a_1)} f(\gamma)$, and $e^{2\pi i/(a_0 a_1)}$ is an eigenvalue of $T_{p(x)}$.

The same argument can be repeated using the eigenvalues $e^{2\pi i/a_0}$ and $e^{2\pi i/a_1}$. \square

Corollary 5.2. *Let $(X_{p(x)}, T_{p(x)})$ be the Bratteli-Vershik system determined by the polynomial $a_0 + a_1 x$ with a fully-supported, $T_{p(x)}$ -invariant, ergodic probability measure μ . If either a_0 or a_1 is greater than 1, then $T_{p(x)}$ is not weakly mixing.*

Remark 2. The main result of this section is possible because for a degree one polynomial, $p(x) = a_0 + a_1 x$, all the coefficients of $(p(x))^n$ except the coefficients of x^0 and x^{dn} are divisible by $a_0 a_1$. For a polynomial $p(x)$ of degree higher than 1, the coefficients do not necessarily have a common factor; therefore this argument is not sufficient for polynomials of higher degree.

We will now show that the Euler adic is totally ergodic, in other words, it has no roots of unity (other than 1) as eigenvalues.

Lemma 5.3. *Let (X, T) be the Bratteli-Vershik system determined by the Euler graph. Let $\gamma \in X$ be a path such that there is an $n \in \mathbb{N}$ for which $n - k_n(\gamma) + 1 > 1$, $k_{n+1}(\gamma) = k_n(\gamma) + 1$, and γ_n is not the largest edge (with respect to the edge ordering) connecting $(n, k_n(\gamma))$ and $(n+1, k_n(\gamma) + 1)$. Then $d(T^{A(n, k_n(\gamma))}\gamma, \gamma) = 2^{-n}$.*

Proof. For the remainder of the proof let $k = k_n(\gamma)$. Recall that $Y_n(k, 0)$ and $Y_n(k, A(n, k) - 1)$ are respectively the minimal and maximal cylinders into vertex (n, k) . There are $l, m \in \mathbb{N}$ such that $T^l \gamma \in Y_n(k, A(n, k) - 1)$, $\gamma \in T^m(Y_n(k, 0))$, and $l + m + 1 = A(n, k)$.

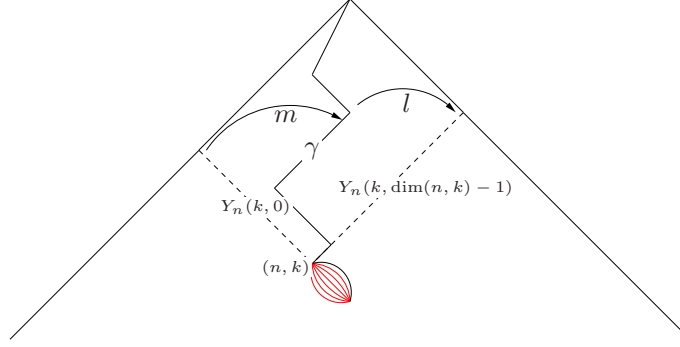


FIGURE 10. γ_n is one of the red paths, and $m + l + 1 = \dim(n, k_n(\gamma))$.

$T(T^l \gamma) \in Y_n(k, 0)$ and $(T^{l+1} \gamma)_n$ is the successor of γ_n with respect to the edge ordering. Then $T^m(T^{l+1} \gamma) = T^{A(n, k)} \gamma \in T^m(Y_n(k), 0)$. Hence

$$(T^{A(n, k)} \gamma)_0 = \gamma_0, (T^{A(n, k)} \gamma)_1 = \gamma_1, \dots, (T^{A(n, k)} \gamma)_{n-1} = \gamma_{n-1}$$

and $(T^{A(n, k)} \gamma)_n \neq \gamma_n$; therefore $d(T^{A(n, k)} \gamma, \gamma) = 2^{-n}$. \square

Key elements of the proof of the following Lemma are already in [?], and a similar result has been known for a long time for substitution and related systems, see [?, ?, ?, ?, ?, ?, ?, ?].

Lemma 5.4. *Let (X, T) be the Bratteli-Vershik system determined by the Euler graph with the symmetric measure η . If λ is an eigenvalue for T , then $\lambda^{A(n, k_n(\gamma))} \rightarrow 1$ η -almost everywhere.*

There is a closed-form formula for $A(n, k)$ which can be found in [?]:

$$A(n, k) = \sum_{j=0}^k (-1)^j \binom{n+2}{j} (k+1-j)^{n+1}.$$

We will apply the following theorem of Lucas to the above equation.

Theorem 5.5 (E. Lucas [?]). *Let p be a prime number and $j \leq n$. Consider the base p decompositions of n and j :*

$$n = n_0 + n_1 p + \dots + n_s p^s$$

$$j = j_0 + j_1 p + \dots + j_s p^s$$

where $0 \leq j_i, n_i < p$ for all i . Then

$$\binom{n}{j} \equiv_p \binom{n_0}{j_0} \dots \binom{n_s}{j_s},$$

with the convention that $\binom{n_i}{j_i} = 0$ if $j_i > n_i$.

Theorem 5.6. *The Bratteli-Vershik system determined by the Euler graph with the symmetric measure η is totally ergodic.*

Proof. It is enough to show that for any prime p , $e^{2\pi i/p}$ is not an eigenvalue for T .

If $\lambda = e^{2\pi i/p}$ is an eigenvalue of T , by Lemma 5.4 we know that $\lambda^{A(n, k_n(\gamma))} \rightarrow 1$ for η -almost every $\gamma \in X$. Since λ is a root of unity, for η -almost every γ in X , there must be an N such that $n \geq N$ implies $\lambda^{A(n, k_n(\gamma))} = 1$. Therefore for η -almost every $\gamma \in X$ there is an N such that $n \geq N$ implies

$$A(n, k_n(\gamma)) \equiv 0 \pmod{p}.$$

We will show that for every $\gamma \in X$, there are infinitely many n for which $A(n, k_n(\gamma)) \equiv_p 1$. In particular, for every $l = 0, 1, \dots$, and $0 \leq k \leq p^l - 1$, $A(p^l - 1, k) \equiv_p 1$. Recall that for $k \geq 1$,

$$A(p^l - 1, k) = \sum_{j=0}^k (-1)^j \binom{p^l + 1}{j} (k + 1 - j)^{p^l}.$$

We will examine this sum by computing each term in the sum mod p . For $j = 0$ we have

$$(5.2) \quad \binom{p^l + 1}{0} (k + 1)^{p^l} = (k + 1)^{p^l} \equiv_p k + 1 \text{ by Fermat's Little Theorem.}$$

For $j = 1$, we have

$$(5.3) \quad (-1) \binom{p^l + 1}{1} k^{p^l} = -(p^l + 1) k^{p^l} \equiv_p -k.$$

For $2 \leq j \leq p^l - 1$ we have

$$(-1)^j \binom{p^l + 1}{j} (k - j)^{p^l}.$$

By Theorem 5.5,

$$(-1)^j \binom{p^l + 1}{j} (k - j)^{p^l} \equiv_p (-1)^j \binom{1}{j_0} \binom{0}{j_1} \dots \binom{0}{j_{l-1}} \binom{1}{0} (k - j)^{p^l}.$$

Since $2 \leq j \leq p^l - 1$, at least one of j_1, j_2, \dots, j_{l-1} must be positive. Therefore

$$(5.4) \quad (-1)^j \binom{p^l + 1}{j} (k - j)^{p^l} \equiv_p 0.$$

For fixed p and $0 \leq k \leq p^l - 1$, we will now compute $A(p^l - 1, k)$.

$$A(p^l - 1, 0) = (-1)^0 \binom{p^l + 1}{0} (1)^{p^l} = 1.$$

Combining Equations 5.2, 5.3, and 5.4 we see that for $k > 0$,

$$A(p^l - 1, k) = \sum_{j=0}^k (-1)^j \binom{p^l + 1}{j} (k + 1 - j)^{p^l} \equiv_p (k + 1) - k \equiv_p 1.$$

Hence $\lambda^{A(n, k_n(\gamma))}$ does not converge to 1, and therefore T has no roots of unity as eigenvalues. \square

The status of weak mixing for both the Euler adic and the Pascal adic is still open.

6. SUBSHIFTS AND ENTROPY

Recall that for systems in $\mathcal{S}_{\mathcal{L}}$ for which $d = 1$, we have that for all $n = 0, 1, \dots$ and $k = 0, 1, \dots, n$, $|V_n| = n + 1$ and there are edges between vertices (n, k) and $(n + 1, k)$ as well as (n, k) and $(n + 1, k + 1)$.

Let $(X, T) \in \mathcal{S}_{\mathcal{L}}$ with $d = 1$. Denote the edges leaving $v_0 = (0, 0)$ by e_1, e_2, \dots, e_m . Define $P_i = \{\gamma \in X \mid \gamma_0 = e_i\}$. Then $\mathcal{P} = \{P_1, P_2, \dots, P_m\}$ is a finite partition of X into pairwise disjoint nonempty clopen cylinder sets. There is a function on X , also denoted by \mathcal{P} such that by $\mathcal{P}(\gamma) = j$ for all $\gamma \in P_j$, $j = 1, \dots, m$. For each $n = 0, 1, 2, \dots$, the \mathcal{P} - n -name of γ is the finite block

$$\mathcal{P}_0^n(\gamma) = \mathcal{P}(\gamma)\mathcal{P}(T\gamma) \dots \mathcal{P}(T^n\gamma),$$

and the \mathcal{P} -name of γ is the doubly infinite sequence

$$\mathcal{P}_{-\infty}^{\infty}(\gamma) = \dots \mathcal{P}(T^{-2}\gamma)\mathcal{P}(T^{-1}\gamma).\mathcal{P}(\gamma)\mathcal{P}(T\gamma)\mathcal{P}(T^2\gamma) \dots$$

For every vertex $(n, k) \in \mathcal{V}$ and $\gamma \in Y_n(k, 0)$ define

$$B(n, k) = \mathcal{P}(\gamma)\mathcal{P}(T\gamma)\mathcal{P}(T^2\gamma) \dots \mathcal{P}(T^{\dim(n, k)-1}\gamma).$$

$B(n, k)$ is called the *basic block at vertex (n, k)* . Let $l(n, k)$ denote the number of edges connecting (n, k) and $(n + 1, k)$, and $r(n, k)$ denote the number of edges connecting (n, k) and $(n + 1, k + 1)$. Then

$$(6.1) \quad B(n + 1, k + 1) = B(n, k)^{r(n, k)} B(n, k + 1)^{l(n, k + 1)},$$

where the exponents indicate concatenation, see Figure 11.

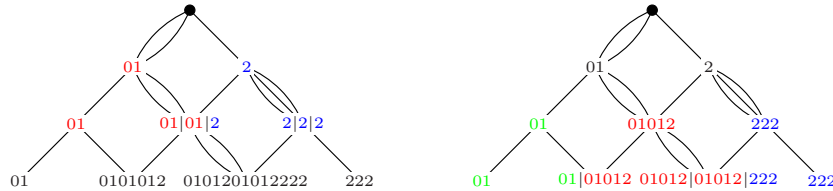


FIGURE 11. Relations of $B(n, k)$ seen graphically.

Definition 6.1. Let Σ denote the space of bi-infinite sequences on $\{1, 2, \dots, m\}$ for which every finite subsequence appears as a subblock in some $B(n, k)$, and let $\sigma : \Sigma \rightarrow \Sigma$ denote the shift map.

We define X' to consist of the maximal set X_{\max} , its orbit, and the set of paths that never leave the far left or far right sides of the diagram:

Lemma 6.2. Define $X' \subset X$ to consist of the following paths:

1. $\mathcal{O}(X_{\max})$;
2. $\{\gamma \in X \mid k_n(\gamma) = 0 \ \forall n \in \mathbb{N}\}$;
3. $\{\gamma \in X \mid k_n(\gamma) = dn \ \forall n \in \mathbb{N}\}$.

Then for any fully supported, T -invariant, ergodic measure μ , $\mu(X') = 0$.

Proof. Since X_{\max} is countable, the sets of paths that never leave the far left ($k_n \equiv 0$) or far right ($k_n \equiv n$) sides of the diagram are proper closed T -invariant sets, and μ is ergodic, $\mu(X') = 0$. \square

Theorem 6.3. *Let $(X, T) \in \mathcal{S}_{\mathcal{L}}$ with $d = 1$, and let μ be a fully-supported T -invariant ergodic probability measure on X . Let Σ be the subshift defined above. Then there are a set $X' \subset X$ with $\mu(X') = 0$ and a one-to-one Borel measurable map $\phi : X \setminus X' \rightarrow \Sigma$ such that $\phi \circ T = \sigma \circ \phi$ on $X \setminus X'$.*

Proof. For each $\gamma \in X$ define $\phi(\gamma)$ to be the \mathcal{P} -name of γ . Then for all $\gamma \in X$,

$$\begin{aligned} \phi \circ T(\gamma) &= \dots \mathcal{P}(T^{-1}\gamma) \mathcal{P}(\gamma) \cdot \mathcal{P}(T\gamma) \mathcal{P}(T^2\gamma) \dots \\ &= \sigma(\dots \mathcal{P}(T^{-2}\gamma) \mathcal{P}(T^{-1}\gamma) \cdot \mathcal{P}(\gamma) \mathcal{P}(T\gamma) \dots) \\ &= \sigma \circ \phi(\gamma). \end{aligned}$$

It is clear that ϕ^{-1} of any cylinder in Σ is a union of cylinder sets in X , hence ϕ is Borel measurable. Defining X' as above, Lemma 6.2 tells us that $\mu(X') = 0$.

The strategy for showing ϕ is one-to-one is to show that for $\gamma, \xi \in X \setminus X'$, $\gamma \neq \xi$, there is a coordinate j such that either $\phi(\gamma)_j$ or $\phi(\xi)_j$ is a symbol from $B(1, 0)$ and the other is a symbol from $B(1, 1)$. This is done in the straightforward but tedious manner of considering cases according to the different ways that $\gamma, \xi \in X \setminus X'$ disagree. We leave the details to the reader. \square

Corollary 6.4. *If (X, T) is a Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$ such that $d = 1$, with fully supported, T -invariant, ergodic measure μ and Borel sets \mathcal{B} , the partition \mathcal{P} is a generating partition.*

While we only know that this partition is generating on systems in $\mathcal{S}_{\mathcal{L}}$ with $d = 1$, we can define such a partition by the first edge for any system in $\mathcal{S}_{\mathcal{L}}$, and define (Σ, σ) as in Definition 6.1.

Lemma 6.5. *For large n the number of words of length n appearing in Σ is bounded above by a polynomial in n (and hence has topological entropy 0).*

Proof. Recall that for each vertex (n, k) in V_n and a path $\gamma \in Y_n(k, 0)$, $B(n, k) = \mathcal{P}(\gamma) \mathcal{P}(T\gamma) \dots \mathcal{P}(T^{\dim(n, k)-1}\gamma)$.

At each level l , we determine the maximum possible number of new words of length n formed by concatenating two words $B(l, k_1)$ and $B(l, k_2)$. The concatenation of $B(l, k_1)$ and $B(l, k_2)$ can form at most $n - 1$ new words. Since there are $dl + 1$ vertices, there are $(dl + 1)^2$ possible distinct concatenations. Hence there are at most $(dl + 1)^2(n - 1)$ new words formed by concatenation.

At level n all blocks except possibly $B(n, 0)$ and $B(n, dn + 1)$ have length at least n . Concatenating $B(n, 0)$ and $B(n, 1)$ creates the word $B(1, 0)^n$. For all levels $m \geq 0$, the edges of the diagrams dictate that $B(m, 0)$ only joins with $B(m, 1)$, hence the concatenation of $B(m, 0)$ and $B(m, 1)$ at levels $m \geq n$ will only create $B(1, 0)$ across their juncture and hence no longer create words that have not been seen before. Likewise for $B(n, dn - 1)$, $B(n, dn)$, and $B(1, d)^n$. All other blocks at level n are of length at least n . Since all words in subsequent levels are created by some concatenations of entries on level n , no more new words are formed.

Therefore the number of words of length n is bounded above by

$$\begin{aligned} \sum_{l=1}^n (dl + 1)^2(n - 1) &\leq n^2(dn + 1)^2 \\ &\leq d^2n^4 + 2dn^3 + n^2. \end{aligned}$$

\square

Theorem 6.6. *Let (X, T) be a Bratteli-Vershik system in $\mathcal{S}_{\mathcal{L}}$ with a T -invariant measure μ . Then (X, T, μ) has entropy 0.*

Proof. One may replace the partition in the proof of Theorem 6.5 by the partition by the first l edges, \mathcal{P}_l , and use the subshift (Σ_l, σ) corresponding to the partition \mathcal{P}_l . A similar counting argument will yield that the number of n -blocks in Σ_l is again bounded by a polynomial in n . Now for each $n = 1, 2, \dots$,

$$\begin{aligned} H_\mu \left(\bigvee_{i=0}^{n-1} T^i \mathcal{P}_l \right) &= - \sum_{A \in \bigvee_{i=0}^{n-1} T^i \mathcal{P}_l} \mu(A) \log(\mu(A)) \\ &\leq H_{ud} \left(\bigvee_{i=0}^{n-1} T^i \mathcal{P}_l \right), \end{aligned}$$

where ud is the measure on $\bigvee_{i=0}^{n-1} T^i \mathcal{P}_l$ that gives each element equal measure. If p_n is the cardinality of $\bigvee_{i=0}^{n-1} T^i \mathcal{P}_l$, then

$$\begin{aligned} H_{ud} \left(\bigvee_{i=0}^{n-1} T^i \mathcal{P}_l \right) &= - \sum_{i=1}^{p_n} \frac{1}{p_n} \log \left(\frac{1}{p_n} \right) \\ &= \log(p_n). \end{aligned}$$

Since there is a constant c_l such that $p_n \leq n^{c_l}$ for all large n , the entropy of the system with respect to the partition \mathcal{P}_l is

$$h_\mu(\mathcal{P}_l, T) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} T^i \mathcal{P}_l \right) \leq \lim_{n \rightarrow \infty} \frac{c_l}{n} \log(n) = 0.$$

Now let \mathcal{B}_l be the σ -algebra generated by \mathcal{P}_l and \mathcal{B} the σ -algebra of (X, T) . Since $\mathcal{B}_l \nearrow \mathcal{B}$, $h_\mu(T)$ is the limit of $h_\mu(\mathcal{P}_l, T)$. Hence the entropy of (X, T, μ) is 0. \square

7. LOOSELY BERNOULLI

The property of loosely Bernoulli was introduced by Feldman in [?] as well as by Katok and Sataev in [?]. A transformation that has zero entropy (see [?]) is loosely Bernoulli if and only if it is isomorphic to an induced map of an irrational rotation on the circle.

Definition 7.1. *The \bar{f} distance between two words $v = v_1 \dots v_l$ and $w = w_1 \dots w_l$ of the same length $l > 0$ on the same alphabet is*

$$\bar{f}(v, w) = \frac{l - s}{l},$$

where s is the greatest integer in $\{0, 1, \dots, l\}$ such that there are $1 \leq i_1 < i_2 < \dots < i_s \leq l$ and $1 \leq j_1 < j_2 < \dots < j_s \leq l$ with $v_{i_r} = w_{j_r}$ for $r = 1, \dots, s$.

Definition 7.2. *Let T be a zero-entropy measure-preserving transformation on the probability space (X, \mathcal{B}, μ) , and let \mathcal{P} be a finite measurable partition of X . Then the process (\mathcal{P}, T) is said to be loosely Bernoulli (LB) if for all $\varepsilon > 0$ and for all sufficiently large l we can find $A \subset X$ with $\mu(A) > 1 - \varepsilon$ such that for all $\gamma, \xi \in A$,*

$$\bar{f}(\mathcal{P}_0^l(\gamma), \mathcal{P}_0^l(\xi)) < \varepsilon.$$

T is said to be loosely Bernoulli if (\mathcal{P}, T) is loosely Bernoulli for all partitions \mathcal{P} .

T is LB if for a generating partition \mathcal{P} , (\mathcal{P}, T) is LB. Some of the Bratteli-Vershik systems determined by positive integer polynomials have already been shown to be loosely Bernoulli. Janvresse and de la Rue proved it for the Pascal adic in [?], and in [?], Méla showed it for polynomials of arbitrary degree where all the coefficients are 1. We have established this property for Bratteli-Vershik systems determined by arbitrary positive integer polynomials as well as for the Euler adic.

Theorem 7.3. *The Bratteli-Vershik systems $(X_{p(x)}, T_{p(x)})$ in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ determined by positive integer polynomials are loosely Bernoulli with respect to each of their $T_{p(x)}$ -invariant ergodic probability measures.*

The proof of Theorem 7.3 will follow the ideas of Janvresse and de la Rue. There are two cases, depending on whether or not the ergodic measure has full support. The following lemma gives a seemingly weaker sufficient condition for the loosely Bernoulli property to hold.

Lemma 7.4 (Janvresse, de la Rue [?]). *Let (X, \mathcal{B}, μ, T) be a measure-preserving system with entropy 0. Suppose that for every $\varepsilon > 0$ and for $\mu \times \mu$ -almost every $(\gamma, \xi) \in X \times X$ we can find an integer $l(\gamma, \xi) \geq 1$ such that*

$$\bar{f}\left(\mathcal{P}_0^{l(\gamma, \xi)}(\gamma), \mathcal{P}_0^{l(\gamma, \xi)}(\xi)\right) < \varepsilon.$$

Then the process (\mathcal{P}, T) is LB.

Lemma 7.5. *Let $(X_{p(x)}, T_{p(x)})$ be the Bratteli-Vershik system determined by the positive integer polynomial $p(x)$ of degree d , with ergodic, $T_{p(x)}$ -invariant probability measure μ . For $\mu \times \mu$ -almost every (γ, ξ) in $X_{p(x)} \times X_{p(x)}$, we can find arbitrarily large n such that $k_n(\gamma) = k_n(\xi)$.*

Proof. Define the random variables $\{Z_i\}_{i=1}^n$ from $X_{p(x)}$ to $\{0, 1, \dots, d\}$ by letting $Z_i(\gamma) = k_i(\gamma) - k_{i-1}(\gamma)$. This is an i.i.d. process. Define $S_n : X_{p(x)} \times X_{p(x)} \rightarrow \{-d, -d+1, \dots, 0, 1, \dots, d\}$ by $S_n(\gamma, \xi) = \sum_{i=1}^n [Z_i(\gamma) - Z_i(\xi)]$. (S_n) is a symmetric random walk, and hence recurrent. Thus for $\mu \times \mu$ -almost every (γ, ξ) in $X_{p(x)} \times X_{p(x)}$, there are infinitely many n such that $k_n(\gamma) = k_n(\xi)$. \square

Proposition 7.6. *Let $(X_{p(x)}, T_{p(x)})$ be a Bratteli-Vershik system determined by a positive integer polynomial of degree d , with a fully-supported, $T_{p(x)}$ -invariant, ergodic probability measure μ . Let \mathcal{P} be the partition determined by the first edge. Then the process $(\mathcal{P}, T_{p(x)})$ is loosely Bernoulli.*

Proof. The partition \mathcal{P} was described in detail in Section 6 for polynomials of degree 1. The same notations will be used here. In particular, recall that for each vertex (n, k) , and a path $\gamma \in Y_n(k, 0)$,

$$B(n, k) = \mathcal{P}(\gamma) \mathcal{P}(T_{p(x)}\gamma) \dots \mathcal{P}(T_{p(x)}^{\dim(n, k)-1} \gamma).$$

From Lemma 7.5, we know that for $\mu \times \mu$ -almost every $(\gamma, \xi) \in X_{p(x)} \times X_{p(x)}$, γ and ξ meet infinitely often. Also, for μ -almost every γ , for each cylinder C ,

$$\frac{\dim(C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))} \rightarrow \mu(C).$$

Let $p_0 = \mu([0])$ denote the weight associated to each edge labeled 0. For each $r \geq 1$, with probability $p_0^{2r} > 0$, both γ and ξ continue along edges labeled 0 for the next r edges.

Given $\varepsilon > 0$, choose r so that $p_0^r < \varepsilon/2$. Let C be a cylinder with terminal vertex (r, dr) . Then for $(\mu \times \mu)$ -almost every (γ, ξ) there are infinitely many n for which $k_n(\gamma) = k_n(\xi) = k$, $\left| \frac{\dim(C, (n, k_n(\gamma)))}{\dim(n, k_n(\gamma))} - \mu(C) \right| < \frac{\varepsilon}{2}$, and $\gamma_n = \xi_n = \gamma_{n+1} = \xi_{n+1} = \dots = \gamma_{n+r-1} = \xi_{n+r-1} = 0$. Then the \mathcal{P} -names of both γ and ξ have long central block $B(n+r, k+dr)$. If we decompose $B(n+r, k+dr)$ into blocks from level n , we see that the first block to appear is $B(n, k)$. Both γ and ξ have their decimal point in this first block of $B(n, k)$.

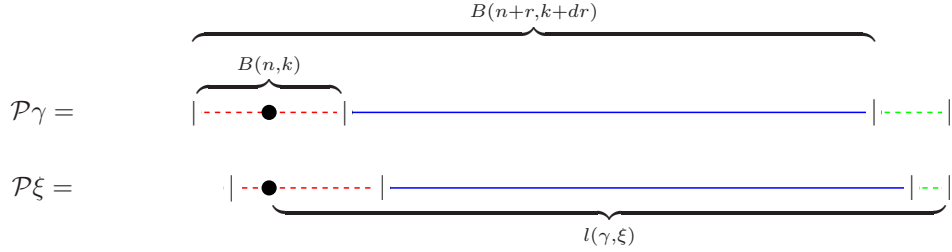


FIGURE 12. $\mathcal{P}\gamma$ and $\mathcal{P}\xi$ agree on the blue line, which is the rest of $B(n+r, k+dr)$ after the end of the initial $B(n, k)$.

If we let $l(\gamma, \xi) = |B(n+r, k+dr)|$ we have

$$\bar{f}(\mathcal{P}_0^{l(\gamma, \xi)} \gamma, \mathcal{P}_0^{l(\gamma, \xi)} \xi) \leq \frac{|B(n, k)|}{|B(n+r, k+dr)|} = \frac{\dim(n, k)}{\dim(n+r, k+dr)}.$$

By the isotropic nature of the diagram, $\dim(n, k) = \dim(C, (n+r, k+dr))$. Hence,

$$\frac{|B(n, k)|}{|B(n+r, k+dr)|} = \frac{\dim(n, k)}{\dim(n+r, k+dr)} = \frac{\dim(C, (n+r, k+dr))}{\dim(n+r, k+dr)}.$$

By Lemma 4.4,

$$\frac{\dim(C, (n+r, k+dr))}{\dim(n+r, k+dr)} \rightarrow \mu(C) = p_0^r.$$

We can now take n large enough so that

$$\frac{|B(n, k)|}{|B(n+r, k+dr)|} < p_0^r + \frac{\varepsilon}{2} < \varepsilon.$$

Hence $(\mathcal{P}, T_{p(x)})$ is loosely Bernoulli. \square

If the partition by the first edge is a generating partition, then Proposition 7.6 would be enough to say that all Bratteli-Vershik systems in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ are loosely Bernoulli. As it stands, we are able to prove that all the systems in $(\mathcal{S}_{\mathcal{L}})_{p(x)}$ are loosely Bernoulli without proving that the partition by the first edge is generating.

Corollary 7.7. *Let $(X_{p(x)}, T_{p(x)})$ be a Bratteli-Vershik system determined by a positive integer polynomial $p(x)$ of degree d with fully-supported, $T_{p(x)}$ -invariant, ergodic probability measure μ . Let \mathcal{P}_l be the partition determined by the first l edges. Then the process $(\mathcal{P}_l, T_{p(x)})$ is loosely Bernoulli.*

Proof. By telescoping to the levels which are multiples of l , the Bratteli-Vershik system becomes the system determined by the polynomial $q(x) = (p(x))^l$, and \mathcal{P}_l becomes the partition on the first edge. By Proposition 7.6, this system is LB. \square

Theorem 7.8 (Ornstein, Rudolph, Weiss [?]). *If G is a compact group and $\phi : G \rightarrow G$ is rotation by ρ with \mathbb{Z}_ρ dense in G , then for any partition \mathcal{P} , (\mathcal{P}, ϕ) is LB.*

Theorem 7.9 (Ornstein, Rudolph, Weiss [?]). *If $\mathcal{B}_n \nearrow \mathcal{B}$ and (X, \mathcal{B}_n, T) is LB for each n , so is (X, \mathcal{B}, T) .*

Proof of Theorem 7.3. If μ does not have full support, by Proposition 4.2, the Bratteli-Vershik system is a stationary odometer. Every odometer is a compact group rotation, and hence by Theorem 7.8 is LB.

For a fully-supported measure μ , Corollary 7.7 says that for each l , the process $(\mathcal{P}_l, T_{p(x)})$ is LB. Let \mathcal{B}_l be the σ -algebra generated by \mathcal{P}_l . Then $(X_{p(x)}, \mathcal{B}_l, \mu, T_{p(x)})$ is LB and $\mathcal{B}_l \nearrow \mathcal{B}$. Hence Theorem 7.9 tells us that $(X_{p(x)}, \mathcal{B}, \mu, T_{p(x)})$ is LB. \square

We now give the same result for the Euler adic.

Theorem 7.10. *Let (X, T) be the Bratteli-Vershik system in $\mathcal{S}_\mathcal{L}$ determined by the Euler graph and with the symmetric measure η . Then T is loosely Bernoulli.*

Proof. Let \mathcal{P} be the partition according to the first edge, described in Section 6 for Bratteli-Vershik systems in $\mathcal{S}_\mathcal{L}$ for which $d = 1$. By Corollary 6.4 this is a generating partition; therefore it is sufficient to show that the process (\mathcal{P}, T) is LB.

Proposition 2 in [?] tells us that for $\eta \times \eta$ -almost every $(\gamma, \xi) \in X \times X$, $k_n(\gamma) = k_n(\xi) = k$ infinitely many times. For such an n , with conditional probability

$$\left(\frac{n - k + 1}{2n + 2} \right)^2,$$

$k_{n+1}(\gamma) = k_n(\gamma) + 1 = k_{n+1}(\xi)$ and both γ_n and ξ_n are one of the first $(n - k + 1)/2$ edges into $(n + 1, k + 1)$. Lemma 2 in [?] tells us that $k_n(\gamma)/n \rightarrow 1/2$ η -almost everywhere. Therefore, for η -almost every $\gamma \in X$,

$$\frac{n - k_n(\gamma) + 1}{2n + 2} \rightarrow \frac{1}{4}.$$

Then for η -almost every $\gamma \in X$ we can take n large enough so that

$$\left(\frac{n - k_n(\gamma) + 1}{2n + 2} \right)^2 > \frac{1}{64}.$$

Hence the set of (γ, ξ) for which there are infinitely many n such that $k_n(\gamma) = k_n(\xi) = k$, $k_{n+1}(\gamma) = k_{n+1}(\xi) = k + 1$, and each of γ_n and ξ_n are one of the first $(n - k + 1)/2$ edges connecting (n, k) and $(n + 1, k + 1)$ has full measure.

Then the \mathcal{P} -names of both γ and ξ have long central block $B(n + 1, k + 1) = B(n, k)^{n-k+1} B(n, k + 1)^{k+2}$. Both γ and ξ have their decimal point in this first block of $B(n, k)$. For some subblocks $w_0 w_1 \dots w_{j_1}$ and $w'_0 w'_1 \dots w'_{j_2}$ of $B(n, k)$ and m_1, m_2 with $(n - k + 1)/2 \leq m_1, m_2 \leq n - k + 1$,

$$\begin{aligned} \mathcal{P}_0^\infty \gamma &= \dots \bullet w_0 w_1 \dots w_{j_1} (B(n, k))^{m_1} (B(n, k + 1))^{k+2} \dots \\ \mathcal{P}_0^\infty \xi &= \dots \bullet w'_0 w'_1 \dots w'_{j_2} (B(n, k))^{m_2} (B(n, k + 1))^{k+2} \dots \end{aligned}$$

Then $\mathcal{P}_0^\infty \gamma$ and $\mathcal{P}_0^\infty \xi$ agree on $\min\{m_1, m_2\}$ consecutive blocks $B(n, k)$. Let $l(\gamma, \xi) = \min\{m_1, m_2\} A(n, k) + \max\{j_1, j_2\}$. Then

$$\overline{f}(\mathcal{P}_0^{l(\gamma, \xi)}(\gamma), \mathcal{P}_0^{l(\gamma, \xi)}(\xi)) \leq \frac{\max\{j_1, j_2\}}{\min\{m_1, m_2\} |B(n, k)|} = \frac{2A(n, k)}{(n - k + 1)A(n, k)} = \frac{2}{n - k + 1}.$$

SARAH BAILEY FRICK THE OHIO STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS, 100 MATH TOWER, 231 W. 18TH AVE

Lemma 2 in [?] says that $k_n(\gamma)/n \rightarrow 1/2$ as $n \rightarrow \infty$. Thus given $\varepsilon > 0$, we can let n be large enough so that

$$\frac{2}{n - k + 1} < \varepsilon.$$

Then T is LB. □

E-mail address: FRICK@MATH.OHIO-STATE.EDU